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Analysis of the Performance Measures of a Non-Markovian Fuzzy Queue via Fuzzy Laplace Transforms Method

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HIGHLIGHTS

- Fuzzy Laplace Transforms applied to analyze performance measures.
- A non-markovian fuzzy queuing system FM/FG/1.
- Queuing systems models play an important role in computers systems implementation
- The performance measures of a non-markovian queue in a fuzzy environment.
- How to apply fuzzy transforms in the evaluation of the measures.

ABSTRACT

Laplace transforms play an essential role in the analysis of classical non-Markovian queueing systems. The problem addressed here is whether the Laplace transform approach is still valid for determining the characteristics of such a system in a fuzzy environment. In this paper, fuzzy Laplace transforms are applied to analyze the performance measures of a non-Markovian fuzzy queueing system FM/ FG/1. Starting from the fuzzy Laplace transform of the service time distribution, we define the fuzzy Laplace transform of the distribution of the dwell time of a customer in the system. By applying the properties of the moments of this distribution, the derivative of this fuzzy transform makes it possible to obtain a fuzzy expression of the average duration of stay of a customer in the system. This expression is the fuzzy formula of the same performance measure that can be obtained from its classical formula by the Zadeh extension principle. The fuzzy queue FM/ FE_k /1 is particularly treated in this text as a concrete case through its service time distribution. In addition to the fuzzy arithmetic of L-R type fuzzy numbers, based on the secant approximation, the properties of the moments of a random variable and Little's formula are used to compute the different performance measures of the system. A numerical example was successfully processed to validate this approach. The results obtained show that the modal values of the performance measures of a non-Markovian fuzzy queueing system are equal to the performance measures of the corresponding classical model computable by the Pollaczeck-Khintchine method. The fuzzy Laplace transforms approach is therefore applicable in the analysis of a fuzzy FM/FG/1 queueing system in the same way as the classical M/G/1 model.

Keywords: *Fuzzy Laplace Transforms, Erlang-k Service, Fuzzy Arithmetic, Performance Measures, Defuzzification.*

INTRODUCTION

Today, no one is unaware of the crucial role played by the modelling of queueing systems in the implementation of computer systems, telecommunication systems, etc.
A queuing system is often described as a circuit in which a client who, on arrival, finds the server(s) busy, decides:

- Or wait for a random period until it is served to leave the circuit;
- Or to go away and come back to request the service after a random time;
- Or to leave once and for all.

Several researches have already successfully analysed different types of classical queues and derived performance measures (Babu, P. S., Kumar, K. S., & Chandan, K. (2022).).

When the descriptor parameters of the system are vague and imprecise, it is called a fuzzy *queueing system*, usually represented by the letter F ***This is usually represented by the letter "Fuzzy***".

Much work has already been done to analyse these types of queues based on fuzzy set theory. This is particularly the case for Markovian fuzzy queuing systems F M/ F M/c and the product-form fuzzy queueing network. The main results of this work can be found in (Fatoumata, Y., Adnane, A., & Ataoua, Z. (2021).).
As for non-Markovian fuzzy models such as F M/ F G/1, F G/ F M/1 ..., the literature is not yet sufficiently extensive to our humble knowledge. Among these rare works, we quote for example of which would be the most recent to our humble knowledge (Al-Kridi, K., Anan, M. T., & Zeina, M. B. (2018).).

Most of these researchers have based their analyses on the method of mathematical optimization programs PNLP (*Parametric Non-Linear Programming*), combining both *the Zadeh Extension Principle* and *the Arithmetic of* α*-and intervals* (Patel, K. R., & Desai, N. B. (2017).).

Others have applied and shown that the L-R method is the fastest and most flexible method to analyze the FM/FEk/1 model (Çitil, H. G. (2019).).
In this article, we asked the question of how to calculate the performance measures of a fuzzy waiting system F M/ F G/1 using *the fuzzy Laplace transform approach* in steady state.

Our hypothesis is that the Laplace transform method would remain valid for analyzing both classical and fuzzy non-Markovian queue performance measures.
Like the classical model, our methodology consists in calculating these measures from the fuzzy Laplace transforms of the distribution of the residence (waiting) times of a customer in the system (the queue) (Gong, Z., & Hao, Y. (2019).).

This text is organized as follows: Section 2 covers the preliminaries, Section 3 deals with fuzzy Laplace transforms starting with the notion of fuzzy functions. Section 4 is devoted to the fuzzy model F M/ F G/1. The concrete case where $G = E_k$ is discussed with a numerical example to validate the method. Finally, section 5 is reserved for the conclusion that ends the text (Chen, G., Liu, Z., & Zhang, J. (2020).).

PRELIMINARIES

Basic concepts

Definition 1 (Fazlollahtabar, H., & Gholizadeh, H. (2019).): Let X be a classical set called universe. A fuzzy subset $\tilde{A}$ of X is defined by a membership function $\mu_{\tilde{A}}$ of X in $[0,1]$ such that:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & if\ x \notin A\ \ (not\ at\ all) \\ r \in]0,1[& if\ x \in A\ (partially) \\ 1 & if\ x \in A\ \ (totally) \end{cases} \quad (1)$$

The essential characteristics of a fuzzy subset $\tilde{A}$ are the α-cuts $\tilde{A}_\alpha$the support $supp(\tilde{A})$the height $h(\tilde{A})$ and the core $noy(\tilde{A})$ defined as follows:

$$\tilde{A}_\alpha = \{x \in X, \mu_{\tilde{A}}(x) \geq \alpha\} \quad (2)$$
$$supp(\tilde{A}) = \{x \in X, \mu_{\tilde{A}}(x) > 0\} \quad (3)$$
$$h(\tilde{A}) = \{\mu_{\tilde{A}}(x), x \in X\} \quad (4)$$
$$noy(\tilde{A}) = \{x \in X, \mu_{\tilde{A}}(x) = 1\} \quad (5)$$

The $\tilde{A}_\alpha$ are also called *parametric representations* of $\tilde{A}$.

Definition 2 (Fazlollahtabar, H., & Gholizadeh, H. (2019).). : A fuzzy subset $\tilde{A}$ is said to be normal if $h(\tilde{A}) = 1$;
$\tilde{A}$ is convex if $\forall x, y \in X, \forall \lambda \in [0,1]$,

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda y)) \geq \{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\} \ ;$$

$\tilde{A}$ is a fuzzy number if $\tilde{A}$ is a fuzzy subset of R such that $noy(\tilde{A}) \neq \emptyset$, $supp(\tilde{A})$ is bounded and $\tilde{A}_\alpha$ are bounded intervals of R. The set of all fuzzy numbers is denoted $F(R)$.

Definition 3: Any real x such that $\mu_{\tilde{A}}(x) = 1$ is said to be a *modal value* or *mode* or the *average value* of the fuzzy number $\tilde{A}$.

A fuzzy number $\tilde{A}$ is said to be strictly positive if $\forall x < 0$, $\mu_{\tilde{A}}(x) = 0$. It is strictly negative if $\forall x > 0$, $\mu_{\tilde{A}}(x) = 0$.

Definition 4 (Gong, Z., & Hao, Y. (2019).). : A fuzzy number $\tilde{A}$ is said to be *triangular* if there are three real numbers $a < b < c$ such that :

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}\ , & if\ \ a \leq x \leq b \\ \frac{c-x}{c-b}\ , & if\ \ \ b < x \leq c \\ 0\ , & otherwise \end{cases} \quad (6)$$

Notation : $\tilde{A} = (a, b, c)$ or $\tilde{A} = (a/b/c)$.

Definition 5 (Gong, Z., & Hao, Y. (2019).). : A fuzzy number $\tilde{A}$ is said to be of *type L-R* if there are three real $m,\ a > 0,\ b > 0$ and two positive, continuous and decreasing functions L and R of R in $[0,1]$ such that :
$L(0) = R(0) = 1$; $L(1) = 0$ or $L(x) > 0\ \forall x \in R$ with $L(x) = 0$;
$R(1) = 0$ or $R(x) > 0\ \forall x \in R$ with $R(x) = 0$;

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{a}\right), & if\ \ x \in [m-a, m] \\ R\left(\frac{x-m}{b}\right), & if\ \ \ x \in [m, m+b] \\ 0\ , & otherwise \end{cases} \quad (7)$$

Notation : $\tilde{A} = \langle m, a, b \rangle_{LR}$ or simply $\tilde{A} = (m, a, b)_{LR}$
Moreover, any triangular fuzzy number $\tilde{A} = (a, b, c)$ is a fuzzy number of type L-R. Its L-R writing is : $\tilde{A} = (a, b, c) = \langle b, b-a, c-b \rangle_{LR}$.

The family of all fuzzy numbers of type L-R is denoted $F_{LR}(R)$ and any real r also called *fuzzy singleton* has the form L-R $r = (r, 0, 0)_{LR}$ (*by convention*).

Fuzzy arithmetic

The algebraic operations on fuzzy numbers, used in this text, are based on three main arithmetics: *the Zadeh extension principle, the arithmetic of α-cuts and intervals* and *the arithmetic of fuzzy numbers of the same type L-R* (Gong, Z., & Hao, Y. (2019).).

The Zadeh extension principle

The arithmetic of the extension principle allows any classical binary operation to be extended $*$ in R to a fuzzy binary operation $\circledast$ in $F(R)$ defined $\forall \tilde{A}, \tilde{B} \in F(R)$, $\forall z \in R$ by (Chen, G., Liu, Z., & Zhang, J. (2020).). :

$$\mu_{\tilde{A} \circledast \tilde{B}}(z) = sup\ sup \left\{\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\} \ / \ x, y \in R, x * y = z\right\} \tag{8}$$

It is defined as follows:

Definition 6 (Fazlollahtabar, H., & Gholizadeh, H. (2019).). : Let $E = E_1 \times \ldots \times E_n$ and F be two classical sets. Let f also be an application from E into F. The extension principle is another application $\tilde{f}$ of $\tilde{P}(E)$ in $\tilde{P}(F)$ such that $\forall \tilde{A} \in \tilde{P}(E), \exists \tilde{B} \in \tilde{P}(F) : \tilde{f}(\tilde{A}) = \tilde{B}$ and $\forall y \in F$we have

$$\{\mu_{\tilde{B}}(y) = sup_{x \in \frac{E}{f(x)} = y} \left\{\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_n}(x_n)\}\right\}\ if\ f^{-1}(y) \neq \emptyset \ \ \mu_{\tilde{B}}(y) = 0\ , \qquad otherwise \tag{9}$$

where f^{-1} is the reciprocal of f and $\tilde{P}(E)$, $\tilde{P}(F)$ are respectively the sets of all fuzzy subsets of E and F.

Arithmetic of α-cuts and intervals

The arithmetic of α-The arithmetic of the cuts is based on the interval arithmetic as defined below:
Definition 7: Let $[a, b]$, $[c, d]$ two bounded real intervals and $*$ the classical operation of addition, subtraction, multiplication or division. We have :

$$[a, b] * [c, d] = [\alpha, \beta] \tag{10}$$

where $[\alpha, \beta] = \left\{x * \frac{y}{a} \leq x \leq b, c \leq y \leq d\right\}$ assuming that $0 \notin [c, d]$ for the division.
In concrete terms, we have :

$$[a, b] + [c, d] = [a + c, b + d] \tag{11}$$

$$[a, b] - [c, d] = [a - d, b - c] \tag{12}$$

$$[a, b] \times [c, d] = [\{ac, ad, bc, bd\}, \{ac, ad, bc, bd\}] \tag{13}$$

$$[a, b] \div [c, d] = \left[\left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\}, \left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\}\right] \tag{14}$$

Definition 8: Let $\tilde{A}, \tilde{B}$ be two fuzzy numbers of respective alpha-slices $\tilde{A}_\alpha = [A^L(\alpha), A^U(\alpha)]$ and $\tilde{B}_\alpha = [B^L(\alpha), B^U(\alpha)]$ $(0 \leq \alpha \leq 1)$. The fuzzy arithmetic operations on $\tilde{A}$ and $\tilde{B}$ are defined via their α-cuts in the following way:

$$\left[\tilde{A} \oplus \tilde{B}\right]_\alpha = \tilde{A}_\alpha + \tilde{B}_\alpha = [A^L(\alpha), A^U(\alpha)] + [B^L(\alpha), B^U(\alpha)] \tag{15}$$

$$\left[\tilde{A} \ominus \tilde{B}\right]_\alpha = \tilde{A}_\alpha - \tilde{B}_\alpha = [A^L(\alpha), A^U(\alpha)] - [B^L(\alpha), B^U(\alpha)] \tag{16}$$

$$\left[\tilde{A} \otimes \tilde{B}\right]_\alpha = \tilde{A}_\alpha \times \tilde{B}_\alpha = [A^L(\alpha), A^U(\alpha)] \times [B^L(\alpha), B^U(\alpha)] \tag{17}$$

$$\left[\tilde{A} \oslash \tilde{B}\right]_\alpha = \tilde{A}_\alpha \div \tilde{B}_\alpha = [A^L(\alpha), A^U(\alpha)] \div [B^L(\alpha), B^U(\alpha)] \tag{18}$$

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These α-These resultant cuts are to be calculated according to the formulas in relations (11) to (14) above (Sanga, S. S., & Jain, M. (2019).).

Arithmetic of fuzzy numbers of the same type L-R

Let $\widetilde{M} = \langle m, a, b\rangle_{LR}$ and $\widetilde{N} = \langle n, c, d\rangle_{LR}$ be two fuzzy numbers of the same L-R type (Sanga, S. S., & Jain, M. (2019).). The arithmetic operations on $\widetilde{M}$ and $\widetilde{N}$ are defined as follows (Babu, P. S., Kumar, K. S., & Chandan, K. (2022).:

$$\widetilde{M} \oplus \widetilde{N} = \langle m+n, a+c, b+d\rangle_{LR} \quad (19)$$

$$\widetilde{M} \ominus \widetilde{N} = \langle m-n, a+d, b+c\rangle_{LR} \quad (20)$$

$$\widetilde{M} \odot \widetilde{N} \approx \{\langle mn, mc+na-ac, md+nb+bd\rangle_{LR} \;, \quad if\ \widetilde{M}, \widetilde{N} > 0\ \langle mn, -md-nb-bd, -mc-na+ac\rangle_{LR}\;, \quad if\ \widetilde{M}, \widetilde{N} < 0\ \langle mn, -md+na+ad, -mc+nb-bc\rangle_{LR}\;, if\ \widetilde{M} < 0, \widetilde{N} > 0 \quad (21)$$

$$\frac{1}{\widetilde{N}} \approx \langle \frac{1}{n}, \frac{d}{n(n+d)}, \frac{c}{n(n-c)}\rangle_{LR}, \widetilde{N} > 0 \quad (22)$$

$$\widetilde{M} \oslash \widetilde{N} \approx \langle \frac{m}{n}, \frac{md}{n(n+d)} + \frac{a}{n} - \frac{ad}{n(n+d)}, \frac{mc}{n(n-c)} + \frac{b}{n} + \frac{bc}{n(n-c)}\rangle_{LR}, \widetilde{M}, \widetilde{N} > 0 \quad (23)$$

Here, multiplication and division are defined by secant approximation rule (Panta, A. P., Ghimire, R. P., Panthi, D., & Pant, S. R. (2021).).

FUZZY TRANSFORMS

Fuzzy functions

There are several types of fuzzy functions: *constraint fuzzy functions, fuzzy functions by propagation of a fuzzy variable and fuzzy functions proper* (Ritha, W., & Rajeswari, N. (2021).).

Definition 9: (Panta, A. P., Ghimire, R. P., Panthi, D., & Pant, S. R. (2021).). : Let X, Y be two universes and $\tilde{P}(Y)$ the set of all fuzzy subsets on Y. The application $\tilde{f} : X \longrightarrow \tilde{P}(Y),\ x \mapsto \tilde{B} = \tilde{f}\,(x)$ is a fuzzy function if

$$\mu_{\tilde{B}}(y) = \mu_{\tilde{R}}(x, y), (\ \forall\ (x, y) \in X{\times}Y\ ) \quad (24)$$

where $\tilde{R}$ is a fuzzy relationship between the elements of $X{\times}Y$.
When $X = [a, b]$, $Y = R$then $\tilde{P}(Y) = F(R)$ and $\tilde{f}$ is a fuzzy function of a real variable. This is the case for the expressions $t \rightarrow \tilde{A}.t + \tilde{B}, t \rightarrow e^{\tilde{a}t}$.

Definition 10 (Wang, F. F. (2022).). : Let $f(t)$ a classical function of variable t. A fuzzy function, an extension of f, is an application denoted $\tilde{f}$ of R in $F(R)$ such that $\tilde{f}(t) = \tilde{Z}$ has as parametric representations the α-cuts :

$$\tilde{Z}_\alpha = [Z^L(\alpha), Z^U(\alpha)] \quad (25)$$

Definition 11 (Wang, F. F. (2022).).: Let $f(x_1, \ldots, x_n)$ a real function of R^n in R and $\tilde{A}_1, \ldots, \tilde{A}_n$ n fuzzy subsets of R. Zadeh's extension principle allows to induce from $f(x_1, \ldots, x_n)$ a fuzzy function $\tilde{f} : F^n(R) \longrightarrow F(R)$ such that $\tilde{f}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ is a fuzzy subset $\tilde{B}$ of R of which :

- The membership function is defined $\forall y \in R | f(x_1, \ldots, x_n) = y$ by :

$$\mu_{\tilde{B}}(y) = \{\left\{\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_n}(x_n)\}\right\} \quad if\ f^{-1}(y) \neq \emptyset\ 0 \qquad if\ f^{-1}(y) = \emptyset \quad (26)$$

- The parametric representation is given $\forall \alpha \in [0, 1]$ by :

$$\tilde{B}(\alpha) = \left(\tilde{f}\left(\tilde{A}_1, \dots, \tilde{A}_n\right)\right)_\alpha = f\left(\tilde{A}_1(\alpha), \dots, \tilde{A}_n(\alpha)\right) \quad (27)$$

This definition establishes the compatibility between the Zadeh extension principle approach and the arithmetic of alpha-slices (Zhang, Q., Sun, H., Gao, X., Wang, X., & Feng, Z. (2022).).
For example, the expressions $\tilde{z} = \frac{2\tilde{x}+10}{3\tilde{x}+4}$ and $\tilde{z} = \frac{\tilde{A}\tilde{x}+\tilde{B}}{\tilde{C}\tilde{x}+\tilde{D}}$ can be considered as fuzzy functions, extensions of the classical functions $h(x) = \frac{2x+10}{3x+4}$ and $g(x_1, x_2, x_3, x_4, x) = \frac{x_1x+x_2}{x_3x+x_4}$.

Fuzzy Laplace Transforms

Definition 12 (Gong, Z., & Hao, Y. (2019).). : Let $\tilde{f}(t)$ a fuzzy function and s a real parameter. The fuzzy Laplace transform of $\tilde{f}(t)$ is a fuzzy function defined by :

$$\hat{F}(s) = L[\tilde{f}(t)] = \int_0^\infty e^{-st}\tilde{f}(t)dt = \int_0^\tau e^{-st}\tilde{f}(t)dt, \quad (28)$$

(insofar as this limit exists).
With respect to its parametric representations, the fuzzy transform of $\tilde{f}(t)$ is written as :

$$\hat{F}(s,\alpha) = L[\tilde{f}(t,\alpha)] = [L[f^L(t,\alpha)], L[f^U(t,\alpha)]], 0 \le \alpha \le 1 \quad (29)$$

where $$\{L[f^L(t,\alpha)] = \int_0^\infty e^{-st}f^L(t,\alpha)dt \; L[f^U(t,\alpha)] = \int_0^\infty e^{-st}f^U(t,\alpha)dt \quad (30)$$

Example: Let $\tilde{a} = (3, 4, 5)$ a triangular fuzzy number and $\tilde{f}(t) = e^{\tilde{a}t}$ a fuzzy function. Then by definition,

$$\hat{F}(s) = \int_0^\infty e^{-st}e^{\tilde{a}t}dt = \frac{1}{s-\tilde{a}}.$$

But,

$$\tilde{a}_\alpha = [3+\alpha, 5-\alpha] \text{ and } \tilde{f}(t,\alpha) = \left[e^{(3+\alpha)t}, e^{(5-\alpha)t}\right].$$

So the representation α-slices of the transform of $\tilde{f}(t) = e^{\tilde{a}t}$ is given by :

$$\hat{F}(s,\alpha) = \left[\int_0^\infty e^{-st}e^{(3+\alpha)t}dt, \int_0^\infty e^{-st}e^{(5-\alpha)t}dt\right] = \left[\frac{1}{s-\alpha-3}, \frac{1}{s+\alpha-5}\right].$$

Properties: Like classical transforms, fuzzy Laplace transforms also have properties such as linearity, translation theorems, derivation theorems, whose eloquent proofs can be found in (Gong, Z., & Hao, Y. (2019).).

FUZZY WAITING SYSTEM $F\ M/\ F\ G/1$

Classical M/G/1 system and Laplace Transforms

The Laplace transform plays a crucial role in the calculation of performance measures of a classical non-Markovian system M/G/1; this is done through the Laplace transform of the general service law G, defined by :

$$B^*(s) = \int_0^\infty e^{-st}b(t)dt, \quad (31)$$

where $b(t)$ is the probability density of this general law.
On the one hand, this Laplace transform $B^*(s)$ transform is used in the definition of the generating function of the stationary probabilities of the system given by (Gong, Z., & Hao, Y. (2019).). :

$$G(z) = (1-\rho)\frac{(1-z)B^*(\lambda-\lambda z)}{B^*(\lambda-\lambda z)-z} \quad (32)$$

where λ and $\rho = \lambda m_1$ are respectively the average arrival rate of customers in the system and the traffic rate in the system (m_1 being the first order moment of the G law).

On the other hand, $B^*(s)$ intervenes in the analysis of the variable W distribution of the residence (and waiting) times of a W_q of a customer in the system (the queue) via its Laplace transform defined by :

$$W^*(s) = (1-\rho)\frac{sB^*(s)}{\lambda B^*(s)-\lambda+s} \quad \text{and} \quad {W_q}^*(s) = (1-\rho)\frac{s}{\lambda B^*(s)-\lambda+s} \tag{33}$$

This allows us to derive the average length of stay and waiting time of a customer as a first order moment of the variable W (variable W_q) :

$$\tau_s = (-1)\frac{dW^*(s)}{ds}(0) \quad \text{and} \quad \tau_q = (-1)\frac{d{W_q}^*(s)}{ds}(0) \tag{34}$$

Other performance measures such as the average number of customers in the system or in the queue are derived by applying Little's law:

$$N_s = \lambda.\tau_s \quad \text{and} \; N_q = \lambda.\tau_q \tag{35}$$

Fuzzy system $F\ M$/ $F\ G$/1 and Fuzzy Laplace Transforms

A fuzzy waiting system is defined as one with vague and imprecise descriptor parameters. In this case, the probability density of the general service law is a fuzzy function of a real variable $\tilde{b}(t)$. Hence the opportunity of its analysis by the fuzzy Laplace transforms, object of this article.

Being then in a fuzzy environment where the system's descriptor parameters are fuzzy, the probability density of the general service law $\tilde{b}(t)$ admits a fuzzy Laplace transform of expression (Gong, Z., & Hao, Y. (2019).). :

$$\tilde{B}^*(s) = \int_0^{\infty} \quad e^{-st}\tilde{b}(t)dt \tag{36}$$

On the one hand, by Zadeh's extension principle, the formulas in relation (33) above become :

$$\widetilde{W}^*(s) = (1-\tilde{\rho})\frac{s\tilde{B}^*(s)}{s-\tilde{\lambda}+\tilde{\lambda}\tilde{B}^*(s)} \quad \text{and} \quad {\widetilde{W}_q}^*(s) = (1-\tilde{\rho})\frac{s}{s-\tilde{\lambda}+\tilde{\lambda}\tilde{B}^*(s)} \tag{37}$$

These are nothing more than fuzzy extensions of the Laplace transforms of the variables W and W_q of the residence and waiting times of a customer in the classical M/G/1 system (in the queue).

On the other hand, although fuzzy, these expressions of relation (37) are functions of a real variable s. Hence the opportunity to exploit the properties of the moments of the variables $\widetilde{W}$ and $\widetilde{W}_q$ to define the average stay and waiting time of a customer in the system (the queue) by the formulas :

$$\tilde{\tau}_s = (-1)\frac{d\widetilde{W}^*(s)}{ds}(0) \;\; \text{and} \;\; \tilde{\tau}_q = (-1)\frac{d{\widetilde{W}_q}^*(s)}{ds}(0) \tag{38}$$

It will therefore be sufficient to apply Little's Law to obtain the other performance measures of the FM/FG/1 system, including :

$$\widetilde{N}_s = \tilde{\lambda}\odot\tilde{\tau}_s \quad \text{and} \quad \widetilde{N}_q = \tilde{\lambda}\odot\tilde{\tau}_q \tag{39}$$

FM/F case studyE_2/1

Position of the problem

As announced above, this case had just been successfully analyzed by *Merlyn Margaret* and her friends (Kannadasan, G., & Sathiyamoorthi, N. (2018).). Using the NLP approach, these authors showed that the problem of analysing a performance measure of a fuzzy expectation system FM/FG/1 can be reduced to the solution of a pair of non-linear parametric programs. After solving these PNLPs, they used the mean degree integral scheme defined by the relation below to defuzzify the obtained fuzzy features:

$$\Phi(\tilde{z}) = \frac{\int_0^1 \frac{\alpha}{2}(Z_\alpha^L + Z_\alpha^U)d\alpha}{\int_0^1 \alpha d\alpha} = \int_0^1 \alpha(Z_\alpha^L + Z_\alpha^U)d\alpha \tag{40}$$

It is a question of dealing with this system $FM/FG/1$ by the fuzzy Laplace transform method as described in the previous subsection.

The results obtained are defuzzified by the "*integral centroid*" approach *named "COA method", i.e. "Center of Area"* defined by (Fazlollahtabar, H., & Gholizadeh, H. (2019).). :

$$\left[\int_{supp(\tilde{Z})} x.\mu_{\tilde{Z}}(x)dx\right] \div \left[\int_{supp(\tilde{Z})} \mu_{\tilde{Z}}(x)dx\right] \tag{41}$$

where Z^* is the classical defuzzified value corresponding to the fuzzy result $\tilde{Z}$ and $\mu_{\tilde{Z}}(x)$ is the membership function.

Resolution

In the classical model M/E_2/1model, the density of the service law $Erlang_2$ and its Laplace transform are given (cf. [5], [16] by :

$$b(t) = (2\mu)^2 te^{-2\mu t} \text{ and } B^*(s) = \left(\frac{2\mu}{s+2\mu}\right)^2 \text{ (for } k = 2) \tag{42}$$

All calculations done, this allows to write the relation (33) as follows:

$$W^*(s) = \frac{4\mu(\mu-\lambda)}{s^2+(4\mu-\lambda)s+4\mu(\mu-\lambda)} \text{ and } W_q{}^*(s) = \frac{\mu-\lambda}{\mu}\frac{(s+2\mu)^2}{s^2+(4\mu-\lambda)s+4\mu(\mu-\lambda)} \tag{43}$$

In the FM/FE_k/1 the service density is a fuzzy extension of the function $b(t)$ given by :

$$\tilde{b}(t) = \frac{(k\tilde{\mu})^k}{(k-1)!} t^{k-1} e^{-k\tilde{\mu}t} \tag{44}$$

By a simple integration calculation, we can establish that :

$$\tilde{B}^*(s) = \int_0^\infty e^{-st}\tilde{b}(t)dt = \left(\frac{k\tilde{\mu}}{s+k\tilde{\mu}}\right)^k \tag{45}$$

Or

$$\tilde{B}^*(s) = \left(\frac{2\tilde{\mu}}{s+2\tilde{\mu}}\right)^2 \text{ for } k = 2 \tag{46}$$

Therefore:

$$\widetilde{W}^*(s) = \frac{4\tilde{\mu}(\tilde{\mu}-\tilde{\lambda})}{s^2+(4\tilde{\mu}-\tilde{\lambda})s+4\tilde{\mu}(\tilde{\mu}-\tilde{\lambda})} \text{ and } W_q{}^*(s) = \frac{\tilde{\mu}-\tilde{\lambda}}{\tilde{\mu}}\frac{(s+2\tilde{\mu})^2}{s^2+(4\tilde{\mu}-\tilde{\lambda})s+4\tilde{\mu}(\tilde{\mu}-\tilde{\lambda})} \tag{47}$$

And according to relation (38) above, we have :

$$\tilde{\tau}_s = (-1)\frac{d\widetilde{W}^*(s)}{ds}(0) = \frac{3\tilde{\lambda}}{4\tilde{\mu}(\tilde{\mu}-\tilde{\lambda})} + \frac{1}{\tilde{\mu}} \text{ and } \tilde{\tau}_q = (-1)\frac{d\widetilde{W}_q{}^*(s)}{ds}(0) = \frac{3\tilde{\lambda}}{4\tilde{\mu}(\tilde{\mu}-\tilde{\lambda})} \tag{48}$$

Numerical example

Statement

Consider a waiting system in which customers arrive according to a Poisson process with an imprecise average rate of about 1 customer per minute. Let us also assume that the service time is distributed according to a$Erlang_2$ process with an imprecise mean of about 1/3. Let us determine the average time a customer stays in the system and waits in the queue respectively.

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Resolution procedure

To say that the average service time of$Erlang_2$ is about 1/3 means that the service rate is about 3. Since these two descriptors are vague and imprecise, we will proceed as follows:

1. Represent these parameters by two triangular fuzzy numbers of modes 1 and 3 respectively: $\tilde{\lambda} = \left(\frac{1}{2}, 1, \frac{3}{2}\right)$ and $\tilde{\mu} = (2, 3, 4)$ for example, then write them in L-R form;
2. Apply the formulas in relation (48), according to the fuzzy Laplace transform approach;
3. Use L-R fuzzy number arithmetic (see relations (19) to (23)) to obtain the expected fuzzy results;
4. Defuzzify these results by relation (41) as announced in 4.3.1 above.

Results obtained

1. Fuzzy numbers $\tilde{\lambda} = \left(\frac{1}{2}, 1, \frac{3}{2}\right)$ and $\tilde{\mu} = (2, 3, 4)$ have the form L-R :

$$\tilde{\lambda} = \langle 1, \tfrac{1}{2}, \tfrac{1}{2}\rangle_{LR} \text{ and } \tilde{\mu} = \langle 3, 1, 1\rangle_{LR}$$

2. From the formulas in relation (48),

$$\tilde{\tau}_s = \frac{3\tilde{\lambda}}{4\tilde{\mu}(\tilde{\mu}-\tilde{\lambda})} + \frac{1}{\tilde{\mu}} = \frac{3\langle 1,\frac{1}{2},\frac{1}{2}\rangle_{LR}}{4\langle 3,1,1\rangle_{LR}\left(\langle 3,1,1\rangle_{LR} - \langle 1,\frac{1}{2},\frac{1}{2}\rangle_{LR}\right)} + \frac{1}{\langle 3,1,1\rangle_{LR}} \tag{49}$$

$$\tilde{\tau}_q = \frac{3\tilde{\lambda}}{4\tilde{\mu}(\tilde{\mu}-\tilde{\lambda})} = \frac{3\langle 1,\frac{1}{2},\frac{1}{2}\rangle_{LR}}{4\langle 3,1,1\rangle_{LR}\left(\langle 3,1,1\rangle_{LR} - \langle 1,\frac{1}{2},\frac{1}{2}\rangle_{LR}\right)} \tag{50}$$

But,

$$3\langle 1, \tfrac{1}{2}, \tfrac{1}{2}\rangle_{LR} = \langle 3, \tfrac{3}{2}, \tfrac{3}{2}\rangle_{LR} \,;\, \frac{1}{\langle 3,1,1\rangle_{LR}} \simeq \langle \tfrac{1}{3}, \tfrac{1}{12}, \tfrac{1}{6}\rangle_{LR} \,;$$

$$4\langle 3, 1, 1\rangle_{LR} = \langle 12, 4, 4\rangle_{LR} \text{ and } \langle 3, 1, 1\rangle_{LR} - \langle 1, \tfrac{1}{2}, \tfrac{1}{2}\rangle_{LR} = \langle 2, \tfrac{3}{2}, \tfrac{3}{2}\rangle_{LR}.$$

Thus, according to the secant approximation formula of relations (21) to (23), the results are :

$$4\langle 3, 1, 1\rangle_{LR}\left(\langle 3, 1, 1\rangle_{LR} - \langle 1, \frac{1}{2}, \frac{1}{2}\rangle_{LR}\right) = \langle 12, 4, 4\rangle_{LR} \otimes \langle 2, \frac{3}{2}, \frac{3}{2}\rangle_{LR}$$
$$\simeq \langle 24, 20, 32\rangle_{LR}$$

Hence,

$$\tilde{\tau}_s \simeq \langle \tfrac{1}{8}, \tfrac{9}{56}, 1\rangle_{LR} + \langle \tfrac{1}{3}, \tfrac{1}{12}, \tfrac{1}{6}\rangle_{LR} \simeq \langle \tfrac{11}{24}, \tfrac{41}{168}, \tfrac{7}{6}\rangle_{LR} \tag{51}$$

$$\tilde{\tau}_q = \frac{\langle 3,\frac{3}{2},\frac{3}{2}\rangle_{LR}}{\langle 24,20,32\rangle_{LR}} \simeq \langle \tfrac{1}{8}, \tfrac{9}{56}, 1\rangle_{LR} \tag{52}$$

These approximate results are none other than the triangular fuzzy numbers

$$\tilde{\tau}_s \simeq \left(\frac{3}{14}, \frac{11}{24}, \frac{13}{8}\right) \text{ and } \tilde{\tau}_q \simeq \left(\frac{-1}{28}, \frac{1}{8}, \frac{9}{8}\right) \tag{53}$$

of modal values $\frac{11}{24}$ and $\frac{1}{8}$ (*units of time*) respectively.

3. Applying Little's formula and the secant approximation of the product of two fuzzy numbers of the same type L-R allows us to obtain the other performance measures, namely the average number of customers in the system and in the queue, i.e. $\tilde{N}_s = \tilde{\lambda} \odot \tilde{\tau}_s$ and $\tilde{N}_q = \tilde{\lambda} \odot \tilde{\tau}_q$:

$$\tilde{N}_s = \tilde{\lambda} \odot \tilde{\tau}_s = \langle 1, \tfrac{1}{2}, \tfrac{1}{2}\rangle_{LR} \odot \langle \tfrac{11}{24}, \tfrac{41}{168}, \tfrac{7}{6}\rangle_{LR} \simeq \langle \tfrac{11}{24}, \tfrac{59}{168}, \tfrac{95}{48}\rangle_{LR} \tag{54}$$

$$\tilde{N}_q = \tilde{\lambda} \odot \tilde{\tau}_q = \langle 1, \tfrac{1}{2}, \tfrac{1}{2}\rangle_{LR} \odot \langle \tfrac{1}{8}, \tfrac{9}{56}, 1\rangle_{LR} \simeq \langle \tfrac{1}{8}, \tfrac{1}{7}, \tfrac{25}{16}\rangle_{LR} \tag{55}$$

In triangular writing, we have :

$$\tilde{N}_s \approx \left(\frac{3}{28}, \frac{11}{24}, \frac{39}{16}\right) \quad \text{and} \quad \tilde{N}_q \approx \left(\frac{-1}{56}, \frac{1}{8}, \frac{27}{16}\right) \tag{56}$$

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of modal values $\frac{11}{24}$ and $\frac{1}{8}$ (*customers per unit of time*) respectively.

4. Finally, the defuzzification of the results obtained by relation (41) requires us to first define their membership functions by relation (6), and then to proceed with the various calculations required:

1° For $\tilde{\tau}_s \simeq \left(\frac{3}{14}, \frac{11}{24}, \frac{13}{8}\right)$ the membership function is given by :

$$\mu_{\tilde{\tau}_s}(x) = \{\frac{168x-36}{41}, \ if \ \frac{3}{14} \leq x \leq \frac{11}{24} \ \frac{39-24x}{28}, \ if \ \frac{11}{24} \leq x \leq \frac{13}{8} \ 0, \quad otherwise \tag{57}$$

$$\tau_s{}^* = \left[\int_{\frac{3}{14}}^{\frac{11}{24}} x.\frac{168x-36}{41}dx + \int_{\frac{11}{24}}^{\frac{13}{8}} x.\frac{39-24x}{28}dx\right] \div \left[\int_{\frac{3}{14}}^{\frac{11}{24}} \frac{168x-36}{41}dx + \int_{\frac{11}{24}}^{\frac{13}{8}} \frac{39-24x}{28}dx\right]$$
$$= 0.781 \ unit \ of \ time;$$

2° For $\tilde{\tau}_q \simeq \left(\frac{-1}{28}, \frac{1}{8}, \frac{9}{8}\right)$ we have :

$$\mu_{\tilde{\tau}_q}(x) = \{\frac{56x+2}{9}, \ if \ \frac{-1}{28} \leq x \leq \frac{1}{8} \ \frac{9-8x}{8}, \ if \ \frac{1}{8} \leq x \leq \frac{9}{8} \ 0, \quad otherwise \tag{58}$$

$$\tau_q{}^* = \left[\int_{\frac{-1}{28}}^{\frac{1}{8}} x.\frac{56x+2}{9}dx + \int_{\frac{1}{8}}^{\frac{9}{8}} x.\frac{9-8x}{8}dx\right] \div \left[\int_{\frac{-1}{28}}^{\frac{1}{8}} \frac{56x+2}{9}dx + \int_{\frac{1}{8}}^{\frac{9}{8}} \frac{9-8x}{8}dx\right]$$
$$= 0.405 \ unit \ of \ time;$$

3° For $\tilde{N}_s \approx \left(\frac{3}{28}, \frac{11}{24}, \frac{39}{16}\right)$ the membership function is given by :

$$\mu_{\tilde{N}_s}(x) = \{\frac{168x-18}{59} \ if \ \frac{3}{28} \leq x \leq \frac{11}{24} \ \frac{117-48x}{95} \ if \ \frac{11}{24} \leq x \leq \frac{39}{16} \ 0 \quad otherwise \tag{59}$$

$$N_s{}^* = \left[\int_{\frac{3}{28}}^{\frac{11}{24}} x.\frac{168x-18}{59}dx + \int_{\frac{11}{24}}^{\frac{39}{16}} x.\frac{117-48x}{95}dx\right] \div \left[\int_{\frac{3}{28}}^{\frac{11}{24}} \frac{168x-18}{59}dx + \int_{\frac{11}{24}}^{\frac{39}{16}} \frac{117-48x}{95}dx\right] =$$
$$1.000 \ customer \ per \ unit \ of \ time;$$

or 60 *customers per hour* if the unit of time is the *minute*.

4° For $\tilde{N}_q \approx \left(\frac{-1}{56}, \frac{1}{8}, \frac{27}{16}\right)$ we have :

$$\mu_{\tilde{N}_q}(x) = \{\frac{56x+1}{8}, \ if \ \frac{-1}{56} \leq x \leq \frac{1}{8} \ \frac{27-16x}{25}, \ if \ \frac{1}{8} \leq x \leq \frac{27}{16} \ 0, \quad otherwise \tag{60}$$

$$N_q{}^* = \left[\int_{\frac{-1}{56}}^{\frac{1}{8}} x.\frac{56x+1}{8}dx + \int_{\frac{1}{8}}^{\frac{27}{16}} x.\frac{27-16x}{25}dx\right] \div \left[\int_{\frac{-1}{56}}^{\frac{1}{8}} \frac{56x+1}{8}dx + \int_{\frac{1}{8}}^{\frac{27}{16}} \frac{27-16x}{25}dx\right]$$
$$= 0.598 \ customer \ per \ unit \ of \ time,$$

Or 36 *customers per hour* if the minute is the chosen unit of time.

Discussion

We find that all modal values of the fuzzy results correspond exactly to the performance measures (*average waiting time, average dwell time, average number of customers in the system and in the queue*) of the classical model $M/E_2/1$ which can be obtained by the Pollaczeck-Khintchine formula mentioned above (Babu, P. S., Kumar, K. S., & Chandan, K. (2022).).

As for the defuzzified values, they are all slightly higher than these modes which are performance measures of the classical model. Shouldn't we see the effects of a fuzzy environment on the performance measures of a waiting system?

CONCLUSION

In this paper, we have sought both to answer the question of what happens to the performance measures of a non-Markovian system in a fuzzy environment and to apply fuzzy transforms in the evaluation of these measures.

To achieve this, we used both the L-R arithmetic of triangular fuzzy numbers and especially the Zadeh extension principle to obtain the fuzzy transform of the distribution of the dwell times of a customer in the system (the queue).

The numerical example treated revealed that, when the descriptor parameters of a system are vague and uncertain, the performance measures, which are fuzzy numbers, have as modal values the performance measures of the corresponding classical model.

The Laplace transform method is therefore still applicable in the evaluation of the performance measures of a fuzzy FM/FG/1 queueing system. Will this be the case for a fuzzy FG/ FM/1 queue? This is a question worth considering.

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CONFLICT OF INTEREST DISCLOSURE

All authors declare that they have no conflicts of interest to disclose

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