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Decomposition of a Fuzzy Function by One-Dimensional Fuzzy Multiresolution Analysis

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HIGHLIGHTS

- Demonstration of the existence of fuzzy multi-resolution analyzes for the decomposition of a fuzzy signal
- Obtaining the fuzzy spaces containing the details of the fuzzy signal by the existence of a fuzzy wavelet
- Construction of a fuzzy wavelet
- obtaining a fuzzy orthonormal basis of $L^2([0,1],\beta(R),\mu,\ F(R))$ on which to decompose a fuzzy signal

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ABSTRACT

Signal compression and data compression are techniques for storing and transmitting signals using fewer bits as possible for encoding a complete signal. A good signal compression scheme requires a good signal decomposition scheme. The decomposition of the signal can be done as follows: The signal is split into a low-resolution part, described by a smaller number of samples than the original signal, and a signal difference, which describes the difference between the low-resolution signal and the real coded signal. Our paper deals with the proofs of these properties in a fuzzy environment. The proof of one-dimensional multiresolution analysis is given. The concept of fuzzy wavelets is introduced and as a byproduct a special fuzzy space of details of a signal is given and an orthonormal basis of $L^2([0,1],\beta(R),\mu,F(R))$ *decomposing the fuzzy signal is obtained.*

Keywords: *Fuzzy image, fuzzy multiresolution analyzes, fuzzy basis functions, fuzzy basis Riesz, fuzzy orthonormal basis.*

INTRODUCTION

The one-dimensional multiresolution analysis of $L^2(R)$ is an appropriate tool for wavelet study it, allows in particular, the construction of an orthonormal bases (Mallat, 1999; Meyer, 1987; Daubechies, 1992; Mehra, 2018).
The multiresolution analysis of a sequence of nested and closed subspaces $(V_j)_{j=-\infty,\ldots,+\infty}$ satisfying the following properties:

1) $\forall j \in Z,\ V_j \subset V_{j+1}$.
2) $\forall j \in Z,\ f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$
3) $\forall k \in Z,\ f(t) \in V_0 \Leftrightarrow f(t-k) \in V_0$
4) $\lim_{j \to -\infty} V_j = \bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$.
5) $\lim_{j \to +\infty} V_j = \overline{\bigcup_{j=-\infty}^{+\infty} V_j} = L^2(R)$

Moreover, there exist $\theta \in L^2(R)$ such that $\{\theta(t-n)\}_{n\in Z}$ is a Riesz basis of V_0 .
A function $f \in L^2(R)$ is approximated at any level j of this analysis, and the approximation in V_j is twice finer than in V_{j-1} for every j = - ∞, **….., +∞.**

Problematic

This multiresolution analysis defines f in $L^2(R)$ using an orthonormal basis, as a sum of details.
The paper deals with this analysis in a fuzzy environment.

Methodology

Our methodological scheme follows the following steps:
- Fuzzy multi-resolution analysis ;
- Detail spaces and wavelets ;
- Construction of the fuzzy wavelet ;
- Fuzzy orthonormal bases of $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$.

Interest of the subject

The interest of our work is that it takes into account the fuzzy environment in the signal decomposition by one-dimensional multiresolution analysis in wavelet theory.

Results obtained:

The main result is **multiresolution analysis and fuzzy orthonormal bases of** $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$

Consider an interval [a, b] as a fuzzy universe set.

The fuzzy partition of this universe is given by the fuzzy subsets of the universe [a, b] which admit the properties given in the following definition:

Definition 1.1 (Perfilieva, 2006; Ohlan, 2021; Bloch, 2015 ; Sussner, 2016)

Consider $x_1 < \dots\dots\dots < x_n$ fixed nodes such that $x_0 = a$ and $x_{n+1} = b$ with $n \geq 2$.
Then the fuzzy sets $A_1, \dots\dots, A_n$, of membership functions $A_1(x), \dots\dots\dots, A_n(x)$ defined on [a, b], form a fuzzy partition of [a, b] if they satisfy the following conditions for
k = 1,.........,n :

(1) $A_k : [a, b] \rightarrow [0, 1]$, $A_k(x_k) = 1$;
(2) $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$;
(3) A_k is continuous;
(4) A_k , for k = 2,........,n , increases strictly on $[x_{k-1}, x_k]$ and decreases strictly on $[x_k, x_{k+1}]$ for k = 1,.........., n - 1.
(5) For all $x \in [a, b]$, $\sum_{k=1}^{n} A_k(x) = 1$

And the membership functions that can be identified with the sets $A_1, \dots\dots\dots, A_n$ are called fuzzy basis functions.

Fuzzy multi-resolution analysis

Let $f : [0, 1] \rightarrow \mathrm{F}(R)$ a fuzzy function and $K(R)$ be the set of closed intervals of R
Then α-cuts of f, $f_\alpha = [f]^\alpha \in K(R)$.

Theorem 1.2
There is a sequence of fuzzy sets $\{V_j\}_{j \in Z}$ forming a multi-resolution analysis of
$L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$.

Proof

Consider a sequence $\{V_j\}_{j \in Z}$ in $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$ and ∀ α∊ [0, 1], let $V_j^\alpha = [V_j]^\alpha$ the α – level sets of V_j .
We have: $V_j^\alpha \in K(R)$.
Assume that this sequence of closed intervals is nested and verifies the following properties:

1) $\forall\, j \in Z,\ V_j^\alpha \subset V_{j+1}^\alpha$;
2) $\forall\, j \in Z, \exists\, f : [0,1] \rightarrow \mathrm{F}(R))$ *such that* $f_\alpha(t) \in V_j^\alpha \Leftrightarrow f_\alpha(2t) \in V_{j+1}^\alpha$;
3) $\forall k \in Z,\ f_\alpha(t) \in V_0^\alpha \Leftrightarrow f_\alpha(t-k) \in V_0^\alpha$;
4) $\lim_{j \to -\infty} V_j^\alpha = \bigcap_{j=-\infty}^{+\infty} V_j^\alpha = \{0\}$;
5) $\lim_{j \to +\infty} V_j^\alpha = \overline{\bigcup_{j=-\infty}^{+\infty} V_j^\alpha}$

$\forall$ α∊ [0, 1], we shown in lemma 1.4 the existence of a Riesz basis $\{\theta^{\alpha}(t-n)\}_{n\epsilon Z}$.

Note that j stands for resolution and represents the level of analysis of the function f_α ; the approximation in V_j^α of f_α is twice fine as in V_{j-1}^α but half good as that in V_{j+1}^α .

Define $V_j = \{v \in \mathrm{F}(R) : [v]^\alpha \in V_j^\alpha \}$ (1.1)

Then for $v\epsilon V_j$, we have : $v_\alpha \in V_j^\alpha \subset V_{j+1}^\alpha$.

Therefore, $v_\alpha \in V_{j+1}^\alpha$ and $v\,\epsilon V_{j+1}$.

The choice of v being arbitrary, we have :

1') $V_j \subset V_{j+1}$, $\forall j \in Z$

2') By definition, if $\forall \alpha \in [0,1]$ $f_\alpha(t) \in V_j^\alpha$, then : $f(t) \in V_j$ and by 2), $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$ $\forall j \in Z$.

3') Similarly, if $\forall \alpha \in [0,1]$, $f_\alpha(t) \in V_0^\alpha$, then : $f(t) \in V_0$ and by 3), $f(t) \in V_0 \Leftrightarrow f(t-k) \in V_0$ $\forall k \in Z$.

Note that :

(i) $\left[\bigcap_{j=-N}^{N} V_j\right]^\alpha = \bigcap_{j=-N}^{N} V_j^\alpha$.

(ii) $\left[\bigcup_{j=-N}^{N} V_j\right]^\alpha = \bigcup_{j=-N}^{N} V_j^\alpha$.

5') From (ii), we have :

$$\lim_{N\to+\infty} \bigcup_{j=-N}^{N} V_j^\alpha = \bigcup_{j=-\infty}^{+\infty} V_j^\alpha \text{ and } \lim_{N\to+\infty}\left[\bigcup_{j=-N}^{N} V_j\right]^\alpha = \left[\lim_{N\to\infty} \bigcup_{-N}^{N} V_j\right]^\alpha = \bigcup_{j=-\infty}^{+\infty} V_j^\alpha.$$

Hence, $\lim_{N\to+\infty} \bigcup_{j=-N}^{N} V_j = \bigcup_{j=-\infty}^{+\infty} V_j$.

Since $V_j \subset V_{j+1}$, we have $\lim_{j\to+\infty} V_j = \overline{\bigcup_{j=-\infty}^{+\infty} V_j}$.

4') V_j^α forms decreasing nested intervals when $j \to -\infty$ that is $V_{-(j+1)}^\alpha \subset V_{-j}^\alpha$, so we have :

$$\bigcap_{j=-\infty}^{\infty} V_j^\alpha = \{0\} \textit{ and } \lim_{j\to+\infty} V_j^\alpha = \{0\} = \bigcap_{j=-N}^{N} V_j^\alpha = \left[\lim_{N\to\infty} \bigcap_{-N}^{N} V_j\right]^\alpha .$$

To complete the proof of theorem 1.2, we need to show the existence of a Riesz basis for V_0^α and therefore, by (1.1) a Riesz basis for V_0.

This is done in lemma 1.4

Definition 1.3 (Mallat, 1999 ; Le Cadet, 2004)

A family of vectors $\{e_n\}_{n\in Z}$ is a Riesz basis of H if it is linearly independent and there exist A > 0 and B > 0 such that for any $f \in$ H, we can find $a[n]$ with

$f = \sum_{n=-\infty}^{+\infty} a[n] e_n$ satisfactory $A\| f \|^2 \le \sum_{n=-\infty}^{+\infty} | a[n] |^2 \le B \| f \|^2$.

Note that this energy equivalence ensures that the development of f on $\{e_n\}_{n\in Z}$ is numerically stable.

The following theorem, inspired by (Mallat, 1999), gives a necessary and sufficient condition for $\{ \theta^{\alpha}(t-n) \}_{n\in Z}$ to be a Riesz basis of V_0^{α} .

Lemma 1.4

A family $\{\theta^{\alpha}(t-n\}_{n\in Z}$, α∈ [0, 1], is a Riesz basis of V_0^{α} if and only if

$$\exists\, 0 < A \text{ and } 0 < B \text{ such that } \forall\, w \in [-\pi, \pi],\ \frac{1}{B} \le \sum_{k\in Z} \left| \hat{\theta}^{\alpha}(w+2k\pi) \right|^2 \le \frac{1}{A} \qquad (1.2)$$

Proof

(i) By definition, $\{ \theta^{\alpha}(t-n) \}_{n\in Z}$ is a Riesz basis of V_0^{α} if ∀ f ϵ V_0^{α} ,

$f(t) = \sum_{n\in Z} a[n]\theta^{\alpha}(t-n)$ and there exist A > 0 and B > 0 such that

$$A\| f \|^2 \le \sum_{n\in Z} | a[n] |^2 \le B \| f \|^2 \qquad (1.3)$$

The Fourier transform of f is $\hat{f}(w) = \hat{a}(w)\ \hat{\theta}^{\alpha}(w+2k\pi)$ where $\hat{a}(w) = \sum_{n\in Z} a[n] e^{-i\pi w}$, w ϵ [- π, π].

By the Parseval identity, we have :

$$\sum_{n\in Z} | a[n] |^2 = \frac{1}{2\pi}\int_0^{2\pi} | \hat{a}(w) |^2 dw$$

and

$$\| f \|^2 = | f(t) |^2 dt = \frac{1}{2\pi}\int_0^{2\pi} | \hat{f}(w) |^2 dw.$$

Using the periodicity of $\hat{a}(w)$, we have :

$$\| f \|^2 = \frac{1}{2\pi}\int_0^{2\pi} | \hat{a}(w) |^2 \sum_{k\in Z} \left| \hat{\theta}^{\alpha}(w+2k\pi) \right|^2 dw.$$

And by (1.3), we have : ∀ w ϵ [-π, π] :

$$\| f \|^2 \le B \| f \|^2 \sum_{k\in Z} \left| \hat{\theta}^{\alpha}(w+2k\pi) \right|^2 .$$

Hence $\frac{1}{B} \le \sum_{k\in Z} \left| \hat{\theta}^{\alpha}(w+2k\pi) \right|^2$

Similarly, we have : $A\|f\|^2\sum_{k\in Z}\left|\hat{\theta}^\alpha(w+2k\pi)\right|^2 \le \|f\|^2$ which implies $\sum_{k\in Z}\left|\hat{\theta}^\alpha(w+2k\pi)\right|^2 \le \frac{1}{A}$.

(2i) Conversely, if f verifies (1.2) then $\{\theta^\alpha(t-n)\}_{n,\in Z}$ is a Riesz basis of V_0^α if and only if $\forall f \in V_0^\alpha$ and for any sequence $(a(n))_{n\in Z} \subset l^2$, we have :

$$A\|f\|^2 \le \sum_{n\in Z}|a[n]|^2 \le B\|f\|^2$$

Suppose that for one of these sequences, (1.2) is not verified.
Then $\forall\, w \in[-\pi, \pi]$, $\exists\ \hat{a}(w)$, with support in [-π, π], such that

$$\frac{1}{B} \succ \sum_{k\in Z}\left|\hat{\theta}^\alpha(w+2k\pi)\right|^2 \; or \; \frac{1}{A} \prec \sum_{k\in Z}\left|\hat{\theta}^\alpha(w+2k\pi)\right|^2.$$

Let us first assume that for these $w \in [-\pi, \pi]$, we have $\sum_{k\in Z}\left|\hat{\theta}^\alpha(w+2k\pi)\right|^2 \prec \frac{1}{B}$

So $\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi}|\hat{a}(w)|^2\sum_{k\in Z}\left|\hat{\theta}^\alpha(w+2k\pi)\right|^2 dw.$

$$\prec \frac{1}{B}\frac{1}{2\pi}\int_0^{2\pi}|\hat{a}(w)|^2 dw = \frac{1}{B}\sum_{n\in Z}|a[n]|^2$$, that is $B\|f\|^2 \prec \sum_{n\in Z}|a[n]|^2$.

Assume also that for these $w \in [-\pi, \pi]$, we have : $\frac{1}{A} \prec \sum_{k\in Z}\left|\hat{\theta}^\alpha(w+2k\pi)\right|^2$.

So $\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi}|\hat{a}(w)|^2\sum_{k\in Z}\left|\hat{\theta}^\alpha(w+2k\pi)\right|^2 dw$

$$\Rightarrow \frac{1}{A}\frac{1}{2\pi}\int_0^{2\pi}|\hat{a}(w)|^2 dw \prec \|f\|^2$$

$\Rightarrow \frac{1}{A}\sum_{n\in Z}|a[n]|^2 \prec \|f\|^2$, that is $\sum_{n\in Z}|a[n]|^2 \prec A\|f\|^2$.

By this double contradiction, the reciprocal is well verified.

Detail spaces and wavelets

Definition 1.5 (Beg, 2013; Cheng, 2015 ; Huang, 2016)

Let A_k be a fuzzy basis function and let $\delta_k(x)$ be an other basis function satisfying all the conditions given in Definition 1.1

Then there exists $p \in N$ with $p > 1$ such that $\delta_k(x) = A_k^p(x)$ (1.4)

where $A_k^p(x) = A_k(x)$ ....$A_k(x)$ (p times), and $\delta_k(x)$ is called the fuzzy delta function.

This implies that $\int_{-\infty}^{\infty}\delta_k(x)\,dx \prec\prec \int_{-\infty}^{\infty}A_k(x)\,dx$ (1.5) and $\int_{-\infty}^{\infty} A_k(x)\, dx = 1$ (1.6)

Definition 1.6 (Beg, 2013; Cheng, 2015 ; Huang, 2016)

Let $A_k(x)$ (for k = 0,........., n) be fuzzy basis functions.

$\{A_k(x)\}$ are orthogonal fuzzy if $\int_{-\infty}^{\infty} A_j(x)\, A_k(x)\, dx = \begin{cases} \delta_k(x), & j = k \\ \varepsilon(x), & |j-k| = 1 \\ 0, & otherwise \end{cases}$ (1.7)

where $\varepsilon(x)$ is a function such that $\int_{-\infty}^{\infty} \varepsilon(x)\, dx = \alpha \prec\prec \int_{-\infty}^{\infty} \delta_k(x)\, dx$ (1.8)
where α is an arbitrary positive real number close to 0.

Definition 1.7 (Beg, 2013)

Consider a fuzzy basis function $A(x)$ centered on the first node, that is k = 0.
We define a displacement operator (R_k) as follows:

$$A_k(x) = R_k\, A(x) \qquad (1.9)$$

Definition 1.8 (Beg, 2013)

The fuzzy scalar product is defined by : $\langle A, R_k\, A\rangle = \bigoplus_{m=-\infty}^{\infty} A \otimes A_k$ (1.10)

where $(A \otimes A_k)\,(m) = A(m)\, A_k(m)$ (1.11)
is an ordinary product.
Furthermore, the sum of any 2 terms in (1.10) is calculated as follows:

$$(A \otimes A_k)\,(m) \oplus (A \otimes A_k)\,(n) = (A \otimes A_k)\,(m) + (A \otimes A_k)\,(n) - (A \otimes A_k)\,(m)\,(A \otimes A_k)\,(n) \qquad (1.12)$$

Definition 1.9 (Beg, 2013)

Let $A_k(x) = R_k\, A(x)$ (for k = 0,.........,n) be fuzzy basis functions satisfying the equations (1.6) and (1.7).

Then $\{A_k(x)\}$ are fuzzy orthogonal. This implies : $\langle A(x), R_k\, A(x)\rangle = \begin{cases} \delta_k(x), & k = 0 \\ \varepsilon(x), & |k| = 1 \\ 0, & otherwise \end{cases}$ (1.13)

where $\langle \cdot, \cdot \rangle$ is a scalar product.

as $\int_{-\infty}^{\infty} \varepsilon(x)\, dx = \alpha \prec\prec \int_{-\infty}^{\infty} \delta_k(x)\, dx$, we can approximate $\langle A(x), R_k\, A(x)\rangle$ as follows:

$$\langle A(x), R_k\, A(x)\rangle = \begin{cases} \delta_k(x), & k = 0 \\ 0, & otherwise \end{cases} \qquad (1.14)$$

From this approximation, it is possible to orthogonalize the basis $\{\theta\,(t - n)\}_{n\epsilon Z}$ of V_0 , and obtain an orthonormal basis $\{\Phi(t - n)\}_{n\epsilon Z}$ of V_0 .

Thus, as $\{\Phi(t - n)\}_{n\epsilon Z}$ is an orthonormal basis of V_0 , the properties (2') and (3') of fuzzy multiresolution analysis allow us to deduce that $\left\{\phi_{jn}\right\}_{n\in Z} = \left\{2^{j/2}\ \phi(2^j t - n)\right\}_{n\in Z}$ form a fuzzy orthonormal basis of V_j for any j ϵ Z.

While these bases are suitable for approximation problems, they do not a priori have properties that facilitate the detection of singularities in an image; on the other hand, the details that are lost when going from a resolution j to a coarser resolution j – 1, are high-frequency components of the image.
Let W_{j-1} be the fuzzy space containing these details.

In the following, we define the direct sum between two fuzzy sets by using α-cuts.

Let P_K (R) be the set of compact and convex subsets of R.
It is known that ∀ u ϵ $F(R)$, the α - cut $[u]^\alpha$ ϵ P_K (R), 0 ≤ α ≤ 1.
For every 0 ≤ α ≤ 1 and for every u, v ϵ $F(R)$, we define $u \tilde{+} v$ using α-cuts $[u \tilde{+} v]^\alpha$ as follows:

Lemma 1.10 (Lakshmikantham, 2003; De Barros, 2017; Gomes, 2015 ; Mazandarani, 2021).

Let u and v ϵ $F(R)$, then ∀ α ϵ [0, 1] : $[u \tilde{+} v]^\alpha = [u]^\alpha + [v]^\alpha$.

We can define the direct sum between two fuzzy sets using α-cuts by :

Definition 1.11 (Cognet, 2000; Grifone, 2019)

$[w]^\alpha = \left[u \underset{D}{\oplus} v\right]^\alpha$ where :
$[w]^\alpha = [u]^\alpha + [v]^\alpha \; with \; [u]^\alpha \cap [v]^\alpha = \{0\}.$
As $V_j \subset V_{j+1}$, there is a subset W_j such that $V_{j+1} = V_j \underset{D}{\oplus} W_j$.
We define this relationship using the α-cuts by :

Definition 1.12

$V_j^\alpha = V_{j-1}^\alpha + W_{j-1}^\alpha$ with $V_{j-1}^\alpha \cap W_{j-1}^\alpha = \{0\}.$
The second condition implies orthogonality.

Now we present fuzzy orthonormal bases of these detail spaces; they will have interesting properties for the detection of singularities in an image, and in particular for the compression problem.
According to the definition of a fuzzy multiresolution analysis, we have :

$$V_0 \subset V_1$$

Since $\Phi(t)$ ϵ V_0 , we have $\Phi(t)$ ϵ V_1 ; hence, there exists a sequence $(h_k)_{k\epsilon Z}$ such that :

$$\phi(t) = \sum_{k\in Z} h_k . \sqrt{2}\ \phi(2t - k).$$

Given Φ, this relation allows to construct h_k (via its transfer function m_0 (w), given in equation (1.16)).
On the other hand,

$$W_0 \subset V_1 .$$

If $\Psi(t)$ is a function of W_0 , there exists a sequence $(g_k)_{k\epsilon Z}$ such that :

$$\psi(t) = \sum_{k \in Z} g_k . \sqrt{2}\, \phi(2t - k).$$

This relationship and the previous one are called fuzzy two-scale relationships.
These two relations allow us to construct a fuzzy wavelet Ψ such that $\{\Psi(t - n)\}_{n\epsilon Z}$ be a fuzzy orthonormal basis of W_0 .
By compressing or expanding Ψ, we then construct fuzzy orthonormal bases of the other detail spaces:

$$\{\psi_{jn}\}_{n\in Z} = \left\{ 2^{j/2}\, \psi(2^j t - n) \right\}_{n \in Z} \; is\ a\ fuzzy\ basis\ of\ W_j \ \ for\ \ j \in Z .$$

Construction of Ψ

Definition 1.13 (Kumwimba, 2016; Feng, 2001; Hesamian, 2022; Chachi, 2018)

Let $\tilde{u}$ *and* $\tilde{v} \in \mathrm{F}(R)$.

We define the operator $\langle \bullet , \bullet \rangle : \mathrm{F}(R) \text{ x } \mathrm{F}(R) \rightarrow \overline{R}$ by the equation

$$\langle \tilde{u} , \tilde{v} \rangle = \int_0^1 (\tilde{u}_\alpha^L . \tilde{v}_\alpha^L + \tilde{u}_\alpha^U \tilde{v}_\alpha^U)\, d\alpha \quad \text{for all } \alpha \in [0, 1] \tag{1.15}$$

Thus, the two filters $g = (g_n)_{n\epsilon Z}$ and $h = (h_n)_{n\epsilon Z}$ that appear in the two-scale relations are expressed in terms of Φ and Ψ: it is sufficient to do the scalar product above between each of the two relations and $\sqrt{2}\, \phi(2t - n)$ and to note $\{ \sqrt{2}\, \phi(2t - k) \}_{k \in Z}$ is orthonormal to obtain :

$$h_n = \sqrt{2} \int_0^1 \left[\phi_\alpha^U(t) . \phi_\alpha^U(2t - n) + \phi_\alpha^L(t) . \phi_\alpha^L(2t - n) \right] d\alpha \quad ;$$

$$g_n = \sqrt{2} \int_0^1 \left[\psi_\alpha^U(t) . \phi_\alpha^U(2t - n) + \psi_\alpha^L(t) . \phi_\alpha^L(2t - n) \right] d\alpha$$

Applying the Fourier transform to each of the scaling relationships, we obtain (Meyer, 1987; Daubechies, 1992) the equations :

$$\hat{\phi}(w) = m_0(w/2) . \hat{\phi}(w/2) \tag{1.16}$$

$$\hat{\psi}(w) = m_1(w/2) . \hat{\phi}(w/2) \tag{1.17}$$

where $m_0(w) = \dfrac{1}{\sqrt{2}} \sum_{k \in Z} h_k . e^{-2 i \pi w k}$

$$m_1(w) = \frac{1}{\sqrt{2}} \sum_{k \in Z} g_k . e^{-2 i \pi w k}$$

are the transfer functions of the filters $\dfrac{1}{\sqrt{2}} h$ and $\dfrac{1}{\sqrt{2}} g$.

Let us look for a function Φ that is a smoothing kernel that is $\hat{\phi}(0) = 1$ and reapply (1.16) to $\hat{\phi}\left(w/2\right)$, then to $\hat{\phi}\left(w/4\right)$, and so on.

Finally, we obtain: $\hat{\phi}(w) = \prod_{j=1}^{+\infty} m_0(w/2^j)$.

This makes it possible to express Φ as a function of h in the case where the starting data of the problem is the filter h.
Knowing $m_1(w)$, the expression of the function Ψ in the case where the starting point of the problem is the filter g can be deduced by equation (1.17).

Fuzzy orthonormal bases of $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$.

Theorem 1.14

Let $\ldots\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots\ldots$ be a fuzzy multiresolution analysis of $L^2([0,1], \beta(R), \mu, \mathcal{F}(R))$.
If Ψ is a fuzzy wavelet constructed according to the above procedure, then this wavelet provides a fuzzy orthonormal basis of $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$.

Proof

To do this, it is sufficient to use definition 1.12 on V_j, then on V_{j-1}, ... up to a certain level L to obtain :

$$V_j = V_L \underset{D}{\oplus} W_L \underset{D}{\oplus} W_{L+1} \underset{D}{\oplus} \ldots\ldots\ldots\ldots \underset{D}{\oplus} W_{j-1}.$$

By properties 4') and 5') of the fuzzy multiresolution analysis : $L^2([0,1], \beta(R), \mu, \mathrm{F}(R)) = \overset{+\infty}{\underset{j=-\infty}{\oplus_D}} W_j$ that is: the space $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$ is decomposed as an orthogonal sum of detail spaces at all resolutions.

Consider a fuzzy function f of $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$.
The previous formula allows us to decompose it on the fuzzy orthonormal bases defined on the spaces $(W_j)_{j\epsilon Z}$:

$$f(t) = \sum_{j \in Z} \sum_{k \in Z} d_{j,k}\, \psi_{jk}(t) \quad \text{où} \quad d_{j,k} = \langle f, \psi_{jk} \rangle$$

with the coefficients $(d_{j,k})_{k\epsilon Z}$ corresponding to the wavelet coefficients of f at resolution j
Thus, $\{\Psi_{jk}(t)\}_{j\epsilon Z,\, k\epsilon Z}$ defines a fuzzy orthonormal basis of $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$ on which f is decomposed into a sum of finer and finer details as j increases.
Note, again by properties 4') and 5') of the fuzzy multiresolution analysis, that we also have:

$$L^2([0,1], \beta(R), \mu, \mathrm{F}(R)) = V_L \overset{+\infty}{\underset{j=L}{\oplus_D}} W_j.$$

$f \in L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$ is then decomposed as follows :

$$f(t) = \sum_{k \in Z} c_{L,k}\, \phi_{Lk}(t) + \sum_{j \in Z,\, j \geq L} \sum_{k \in Z} d_{j,k}\, \psi_{jk}(t).$$

$\sum_{k \in Z} c_{L,k}\, \phi_{Lk}$ is the projection of f onto an approximation space V_L, $\sum_{j \in Z,\, j \geq L} \sum_{k \in Z} d_{j,k}\, \psi_{jk}(t)$ contains all the details that were lost when approximating f onto V_L .

Restriction to the bounded interval [0, 1]: periodic fuzzy wavelet bases

Theorem 1.15

Consider a fuzzy multiresolution analysis of $L^2([0,1], \beta([0,1]), \mu, \mathrm{F}([0,1]))$.
Given a fuzzy wavelet Ψ, this wavelet allows us to obtain a fuzzy orthogonal basis of $L^2([0,1], \beta([0,1]), \mu, \mathrm{F}([0,1]))$.

Proof

In fact, since in this case the signals we manipulate are in practice of bounded support: we must define fuzzy wavelet bases on a bounded interval [0, 1].

To define a fuzzy wavelet basis on [0, 1], we start from a basis of
$L^2([0,1], \beta(R), \mu, \mathrm{F}(R)),\ \{\psi_{jn}\}_{j\in Z, n\in Z} = \left\{ 2^{j/2}\ \psi(2^j t - n) \right\}_{j\in Z, n\in Z}$.
The fuzzy wavelets $\Psi_{jn}(t)$ spanning t = 0 or t = 1 will have to be adapted.
The simplest method is to periodise the wavelets Ψ_{jn} and the function f.
To do this, we define :

$$f^{per}(t) = \sum_{k=-\infty}^{+\infty} f(t+k)\ \ et\ \ \psi_{jn}^{per}(t) = \sum_{k=-\infty}^{+\infty} \psi_{jn}(t+k).$$

$\psi_{jn}^{per}\ et\ \ f^{per}$ are periodic, of period 1.

If the support of Ψ_{jn} lies in [0, 1], $\psi_{jn}^{per} = \psi_{jn}$ (and even if the support of the fuzzy wavelet Ψ is not compact, on a small scale, ψ_{jn}^{per} will tend to ψ_{jn}): the behaviour of the fuzzy inner wavelets is not affected.
ϕ_{jn}^{per} is defined in the same way by periodising the fuzzy scale functions.
This gives that for all J ≥ 0, the family
$\left[\{\phi_{J,n}^{per}\}_{n=0,\dots\dots,2^J-1},\ \{\psi_{j,n}^{per}\}_{j\geq J,\ n=0,\dots\dots,2^j-1} \right]$ is a fuzzy orthonormal basis of
$L^2([0,1], \beta([0,1]), \mu, \mathrm{F}\ ([0,1]))$.

The spaces of fuzzy approximations V_j^{per} and the spaces of fuzzy details W_j^{per} are of finite dimensional spaces.

In other words, since $\psi_{jn}^{per}(t+2^j) = \psi_{jn}^{per}(t) = \psi_{j,n+2^j}^{per}(t)$, at resolution j there are only 2^j different fuzzy wavelets.
The same applies to fuzzy scale functions.
Thus, $V_j^{per} = vect\{\phi_{jk}^{per}\}_{k\in Z}$ is in fact finite-dimensional: $\phi_{jk}^{per} = \phi_{j,k+2^j}^{per}$.

Specifically, V_j^{per} is of dimension 2^j.
In particular, V_0 , the coarsest fuzzy approximation space, is of dimension 1: it is the set of constants on [0, 1].

We also have dim $W_j^{per} = 2^j$.
This periodisation method has the advantage of being simple, but it can generate large wavelet coefficients at the edges, if the function f is not itself periodic.

Note, however, that when periodic boundary conditions are used, the notations can be abbreviated by writing V_j rather than V_j^{per}, Ψ_{jk} instead of Ψ_{jk}^{per} ,..........

Discussion

Our results, in particular the definition and the proof of a one-dimensional fuzzy multiresolution analysis, constitute our major and original contribution. It allowed us to perform the decomposition of a fuzzy signal.

CONCLUSION

A good signal compression scheme requires a good signal decomposition scheme. The signal is subdivided into a low-resolution part, which can be described by a smaller number of bits than the original signal, and a signal difference, which describes the difference between the low-resolution signal and the real coded signal. We have seen that, for a fuzzy signal, this decomposition can be obtained by one-dimensional fuzzy multiresolution analysis via the use of α-cuts. This fuzzy multiresolution analysis allowed the definition of the detail spaces as well as the constructions of a fuzzy wavelet and a fuzzy orthonormal basis of the space $L^2([0,1],\beta(R),\mu,\ \mathrm{F}(R))$ on which the signal is decomposed.

CONFLICT OF INTEREST DISCLOSURE

The authors declare no conflict of interest in the subject matter or materials discussed in this manuscript.

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