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Algorithms for the Extension Two-Dimensional Fuzzy Signal via Decomposition and Reconstruction in Fuzzy Wavelets

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HIGHLIGHTS

- Recall of some definitions and main results obtained on the existence of fuzzy multiresolution analysis in the one-dimensional case, for the decomposition of a fuzzy image.
- Presentation of a fast construction algorithm for fuzzy signal analysis and synthesis based on fuzzy multiresolution analysis.
- Extension of the algorithm to 2 dimensions, by the two-dimensional fuzzy multiresolution analysis.

ABSTRACT

The decomposition of an image can be done in the following way: The image is split into a low-resolution part, which can be described by a smaller number of samples than the original image, and a signal difference, which describes the difference between the low-resolution image and the real coded image. Therefore, this low-resolution image is also decomposed into a low-resolution image and a difference image, making more efficient coding possible. This decomposition is repeated several times, so that a hierarchical image decomposition is created. Thus, the low-resolution image is only half the size of the original image. This reduced image is enlarged to the size of the original image. The result is a detailed image that is the same size as the original image. Our problem is: "Can we build algorithms allowing the decomposition and reconstruction of a signal in a fuzzy environment? We will answer in the affirmative.
This construction is first made possible by one-dimensional fuzzy multiresolution analysis, which will later be extended to two dimensions. In the first part, we recall some definitions and main results obtained on the existence of fuzzy multiresolution analysis in the one-dimensional case, for the decomposition of a fuzzy image. The second part, based on this multi-resolution analysis, presents a fast construction algorithm for the analysis and synthesis of a fuzzy signal. Finally, the third part is nothing else than an extension of this algorithm to 2 dimensions, by the fuzzy multiresolution two-dimensional analysis.

Keywords: *fuzzy image, fuzzy basis function, Riesz basis, multi – analysis fuzzy resolution, fuzzy orthonormal basis, orthogonal transform in fuzzy wavelets.*

INTRODUCTION

A good image compression scheme requires a good image decomposition scheme.
The decomposition of the image can be done as follows:
The image is split into a low-resolution part, which can be described by a smaller number of samples than the original image, and a difference image, which describes the difference between the low-resolution image and the real coded image.
Thus, wavelet decomposition reduces the size of the image at each step relative to a given resolution; each of the details of these images being taken into account by the wavelet coefficients, which allow a reconstructed signal to be obtained for its best transmission.

Problematic

The hierarchical decomposition as described above reduces the size of the image from one resolution to another; however, the problem is that it is insufficient because it does not take into account the parameters of the image in an environment containing inaccuracies.

Methodology

Our methodological scheme includes the following steps:

- ✓ Reminder of the definitions and main results on one-dimensional fuzzy multi-resolution analysis;
- ✓ One-dimensional fuzzy wavelet decomposition and reconstruction algorithm ;
- ✓ Generalization of the algorithm to 2 dimensions.

Interest of the subject

The transmission of an image, like any natural phenomenon, involves a large number of uncertainties.
A natural approach to controlling these uncertainties is to consider decomposition and reconstruction in a fuzzy environment.

The interest of this work concerns the taking into account of the fuzzy environment in the construction, by the uni and bidimensional multiresolution analysis, of a wavelet decomposition and reconstruction algorithm

Results obtained

1. Reminder of the definitions and main results on fuzzy multiresolution analysis one dimensional

Definition 1.1 (Perfilieva, 2006; Ohlan, 2021; Bloch, 2015; Sussner, 2016)

Consider $x_1 < \dots\dots\dots < x_n$ fixed nodes such that $x_0 = a$ and $x_{n+1} = b$ with $n \geq 2$.
Then the fuzzy sets A_1 ,......., A_n , of membership functions $A_1(x)$,........., $A_n(x)$ defined on [a, b], form a fuzzy partition of [a, b] if they satisfy the following conditions for k = 1,.........,n :

(1) $A_k : [a, b] \rightarrow [0, 1]$, $A_k(x_k) = 1$;

(2) $A_k(x) = 0$ if $x \notin (x_{k-1}, x_{k+1})$;

(3) A_k is continuous;

(4) A_k , for k = 2,........ .n , grows strictly over [x_{k-1}, x_k] and decreases strictly over [x_k, x_{k+1}] for k = 1,.........., n - 1.

(5) For all $x\in$ [a, b], $\sum_{k=1}^{n} A_k(x)=1$

And the membership functions that can be identified with the sets A_1,.........,A_n are called fuzzy basis functions.

Theorem 1.2

There is a sequence of fuzzy sets $\{V_j\}_{j\in Z}$ forming a multi-resolution analysis of $L^2([0,1],\beta(R),\mu,\ \mathrm{F}(R))$.

Proof

Consider a sequence $\{V_j\}_{j\in Z}$ in $L^2([0,1],\beta(R),\mu,\ \mathrm{F}(R))$ and ∀ αϵ [0, 1], let $V_j^\alpha=[V_j]^\alpha$ *the* $\alpha-$*cuts of* V_j.

We have: $V_j^\alpha\in K(R)$.

Suppose that this sequence of closed intervals is nested and verifies the following properties:

1) $\forall\, j\in Z,\ V_j^\alpha\subset V_{j+1}^\alpha$;
2) $\forall\, j\in Z, \exists\ f:[0,1]\rightarrow \mathcal{F}(R)$ *such that* $f_\alpha(t)\in V_j^\alpha \Leftrightarrow f_\alpha(2t)\in V_{j+1}^\alpha$;
3) $\forall k\in Z,\ f_\alpha(t)\in V_0^\alpha \Leftrightarrow f_\alpha(t-k)\in V_0^\alpha$;
4) $\lim_{j\rightarrow-\infty} V_j^\alpha=\bigcap_{j=-\infty}^{+\infty} V_j^\alpha=\{0\}$;
5) $\lim_{j\rightarrow+\infty} V_j^\alpha=\overline{\bigcup_{j=-\infty}^{+\infty} V_j^\alpha}$

It follows that ∀ α ϵ [0, 1], $\exists\theta\,\epsilon\, L^2([0,1],\beta(R),\mu,\ \mathrm{F}(R))$ such that $\{\theta^\alpha\,(t - n)\}_{n\epsilon\ Z}$ is a Riesz basis of V_0^α .

Note that j stands for resolution and represents the level of analysis of the function f_α ; the approximation in V_j^α of f_α is twice as fine as that in V_{j-1}^α but half as good as that in V_{j+1}^α .

Note that we can define
$V_j=\{v\in \mathrm{F}(R):[v]^\alpha\in V_j^\alpha\}$.

If $v\epsilon\ V_j$, we have : $v_\alpha\in V_j^\alpha\subset V_{j+1}^\alpha$.

Thus, $v_\alpha\in V_{j+1}^\alpha$, that is $v\,\epsilon V_{j+1}$.

Hence, the choice of *v* being arbitrary, we have :

1') $V_j\subset V_{j+1}$, $\forall\, j\in Z$

2') By definition, if $f(t)$ is such that $\forall\alpha\in[0,1]\ f_\alpha(t)\in V_j^\alpha$, we have: $f(t)\in V_j$.

Hence by 2), $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}\ \forall\, j \in Z$.

3') Similarly, if f(t) is such that $\forall \alpha \in [0,1]$, $f_\alpha(t) \in V_0^\alpha$, we have: $f(t) \in V_0$.

Hence by 3), $f(t) \in V_0 \Leftrightarrow f(t-k) \in V_0\ \forall k \in Z$.

Note that :

(i) $\left[\bigcap_{j=-N}^{N} V_j\right]^\alpha = \bigcap_{j=-N}^{N} V_j^\alpha$.

(ii) $\left[\bigcup_{j=-N}^{N} V_j\right]^\alpha = \bigcup_{j=-N}^{N} V_j^\alpha$.

5') From (ii), we have :

On the one hand, $\lim\limits_{N\to+\infty} \bigcup_{j=-N}^{N} V_j^\alpha = \bigcup_{j=-\infty}^{+\infty} V_j^\alpha$

and on the other hand, $\lim\limits_{N\to+\infty} \left[\bigcup_{j=-N}^{N} V_j\right]^\alpha = \left[\lim\limits_{N\to\infty} \bigcup_{-N}^{N} V_j\right]^\alpha = \bigcup_{j=-\infty}^{+\infty} V_j^\alpha$.

Hence: $\lim\limits_{N\to+\infty} \bigcup_{j=-N}^{N} V_j = \bigcup_{j=-\infty}^{+\infty} V_j$.

And since $V_j \subset V_{j+1}$, we deduce that $\lim\limits_{j\to+\infty} V_j = \overline{\bigcup_{j=-\infty}^{+\infty} V_j}$.

4') Since V_j^α forms decreasing nested intervals when $j \to -\infty$ that is

$V_{-(j+1)}^\alpha \subset V_{-j}^\alpha$, we have : $\bigcap_{j=-\infty}^{\infty} V_j^\alpha = \{0\}$.

Hence, $\lim\limits_{j\to+\infty} V_j^\alpha = \{0\} = \bigcap_{j=-N}^{N} V_j^\alpha = \left[\lim\limits_{N\to\infty} \bigcap_{-N}^{N} V_j\right]^\alpha$ by reasoning similar to 5') using (i).

Therefore, we obtain: $\lim\limits_{j\to+\infty} V_j = \bigcap_{j=-\infty}^{\infty} V_j = \chi_{\{0\}}$.

To complete the construction, let's prove that $\{\theta^\alpha(t-n)\}_{n\in Z}$ is a Riesz basis of V_0^α , to deduce that $\{V_j\}$ generates a multiresolution analysis of $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$.

To do this, let us first define a Riesz basis of a space denoted H (Hilbert space).

Definition 1.3 (Mallat, 1999 ; Le Cadet, 2004)

A family of vectors $\{e_n\}_{n\in Z}$ is a Riesz basis of H if it is linearly independent and there exist A > 0 and B > 0 such that for any $f \in$ H, we can find a[n] with

$f = \sum_{n=-\infty}^{+\infty} a[n] e_n$ satisfactory $A\| f \|^2 \le \sum_{n=-\infty}^{+\infty} | a[n] |^2 \le B\| f \|^2$.

Note that this energy equivalence ensures that the development of f on $\{e_n\}_{n\in Z}$ is numerically stable.

The following theorem, inspired by (Mallat, 1999), gives a necessary and sufficient condition for $\{ \theta^\alpha (t-n) \}_{n\in Z}$ to be a Riesz basis of V_0^α .

Theorem 1.4

A family $\{\theta^\alpha (t-n\}_{n\in Z}$, α∈ [0, 1], is a Riesz basis of V_0^α if and only if

$$\exists\ 0 < A \text{ and } 0 < B \text{ such that } \forall\, w \in [-\pi, \pi]\ , \frac{1}{B} \le \sum_{k\in Z} \left| \hat{\theta}^\alpha (w+2k\pi) \right|^2 \le \frac{1}{A} \tag{1}$$

Proof

(i) By definition, $\{ \theta^\alpha (t-n) \}_{n,\in Z}$ is a Riesz basis of V_0^α if $\forall f \in V_0^\alpha$,

$$f(t) = \sum_{n\in Z} a[n] \theta^\alpha (t-n) \text{ and there exist A > 0 and B > 0 such that } A\| f \|^2 \le \sum_{n\in Z} | a[n] |^2 \le B \| f \|^2 \tag{2}$$

The Fourier transform of f is $\hat{f}(w) = \hat{a}(w)\ \hat{\theta}^\alpha (w+2k\pi)$ where $\hat{a}(w) = \sum_{n\in Z} a[n] e^{-i\pi w}$,

$w \in [-\pi, \pi]$.

By the Parseval identity, we have :

$$\sum_{n\in Z} | a[n] |^2 = \frac{1}{2\pi} \int_0^{2\pi} | \hat{a}(w) |^2 dw$$

and

$$\| f \|^2 = | f(t) |^2 dt = \frac{1}{2\pi} \int_0^{2\pi} \left| \hat{f}(w) \right|^2 dw.$$

By exploiting the periodicity of $\hat{a}(w)$, we have :

$$\| f \|^2 = \frac{1}{2\pi} \int_0^{2\pi} | \hat{a}(w) |^2 \sum_{k\in Z} \left| \hat{\theta}^\alpha (w+2k\pi) \right|^2 dw.$$

Using (2), we have: $\forall\, w \in [-\pi, \pi]$ given:

$$\| f \|^2 \le B \| f \|^2 \sum_{k\in Z} \left| \hat{\theta}^\alpha (w+2k\pi) \right|^2 .$$

Hence $\frac{1}{B} \le \sum_{k\in Z} \left| \hat{\theta}^\alpha (w+2k\pi) \right|^2$

Similarly, we have : $A\| f \|^2 \sum_{k\in Z} \left| \hat{\theta}^\alpha (w+2k\pi) \right|^2 \le \| f \|^2$ which implies $\sum_{k\in Z} \left| \hat{\theta}^\alpha (w+2k\pi) \right|^2 \le \frac{1}{A}$.

(2i) Conversely, if f verifies (1) then $\{ \theta^\alpha (t-n) \}_{n,\in Z}$ is a Riesz basis of V_0^α if and only if $\forall f \in V_0^\alpha$ and for any sequence $(a(n))_{n\in Z}$ $a(n) \in l^2$, we have :

$$A\| f \|^2 \le \sum_{n\in Z} | a[n] |^2 \le B \| f \|^2$$

Suppose that for one of these sequences, (1) is not verified.

Then $\forall\, w \in [-\pi, \pi]$, $\exists\, \hat{a}(w)$ whose support is in $[-\pi, \pi]$ such that

$$\frac{1}{B} \succ \sum_{k\in Z} \left| \overset{\wedge}{\theta}^{\alpha}(w+2k\pi) \right|^2 \; or \; \frac{1}{A} \prec \sum_{k\in Z} \left| \overset{\wedge}{\theta}^{\alpha}(w+2k\pi) \right|^2 .$$

Let us first assume that for these $w \in [-\pi, \pi]$, we have $\sum_{k\in Z} \left| \overset{\wedge}{\theta}^{\alpha}(w+2k\pi) \right|^2 \prec \frac{1}{B}$

So $\| f \|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\hat{a}(w)|^2 \sum_{k\in Z} \left| \overset{\wedge}{\theta}^{\alpha}(w+2k\pi) \right|^2 dw.$

$$\prec \frac{1}{B}\frac{1}{2\pi}\int_0^{2\pi} |\hat{a}(w)|^2 = \frac{1}{B}\sum_{n\in Z} |a[n]|^2 \; \text{that is } B\| f \|^2 \sum_{n\in Z} |a[n]|^2 .$$

Assume also that for these $w \in [-\pi, \pi]$, we have : $\frac{1}{A} \prec \sum_{k\in Z} \left| \overset{\wedge}{\theta}^{\alpha}(w+2k\pi) \right|^2 .$

So $\| f \|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\hat{a}(w)|^2 \sum_{k\in Z} \left| \overset{\wedge}{\theta}^{\alpha}(w+2k\pi) \right|^2 dw$

$$\Rightarrow \frac{1}{A}\frac{1}{2\pi}\int_0^{2\pi} |\hat{a}(w)|^2 dw \prec \| f \|^2$$

$$\Rightarrow \frac{1}{A}\sum_{n\in Z} |a[n]|^2 \prec \| f \|^2 \; \text{that is } \sum_{n\in Z} |a[n]|^2 \prec A\| f \|^2 .$$

By this double contradiction, the reciprocal is well verified.

Lemma 1.5 (Lakshmikantham, 2003; De Barros, 2017; Gomes, 2015 ; Mazandarani, 2021).

Consider u and v $\in \mathcal{F}(R)$ (the set of all fuzzy numbers of R).
So $\forall \alpha \in [0,1]: [u \tilde{+} v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}$.

Definition 1.6 (Cognet, 2000; Grifone, 2019)
$w = u \underset{D}{\oplus} v$ is defined by the α-cuts by :
$[w]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}$ as $[u]^{\alpha} \cap [v]^{\alpha} = \{0\}$.

Definition 1.7
Given W_{j-1} the set of details of V_j, $V_j = V_{j-1} \underset{D}{\oplus} W_{j-1}$ such that
$V_j^{\alpha} = V_{j-1}^{\alpha} + W_{j-1}^{\alpha}$ with $V_{j-1}^{\alpha} \cap W_{j-1}^{\alpha} = \{0\}$.

Definition 1.8 (Kumwimba, 2016; Feng, 2001; Hesamian, 2022; Chachi, 2018)

Let $\tilde{u}$ and $\tilde{v} \in \mathrm{F}(R)$.
We define the operator $\langle \bullet, \bullet \rangle : \mathrm{F}(R) \times \mathrm{F}(R) \to \overline{R}$ by the equation

$$\langle \tilde{u}, \tilde{v} \rangle = \int_0^1 \left(\tilde{u}_\alpha^L \ \tilde{v}_\alpha^L + \tilde{u}_\alpha^U \tilde{v}_\alpha^U \right) d\alpha \tag{3}$$

Theorem 1.9

Consider a fuzzy multiresolution analysis of $L^2([0,1], \beta(R), \mu, \mathcal{F}(R)), \ldots.. \subset V_{-1} \subset V_0 \subset V_1 \subset$ ....

If ψ is a fuzzy wavelet associated to this fuzzy multiresolution analysis, then it generates a fuzzy orthonormal basis of $L^2([0,1], \beta(R), \mu, \mathrm{F}(R))$.

In practice signals have bounded support : therefore we must define fuzzy wavelet bases on a bounded interval [0, 1]. In this case, an equivalent result is :

Theorem 1.10

Consider a fuzzy multiresolution analysis of $L^2([0,1], \beta([0,1]), \mu, \mathrm{F}(R))$.
Let ψ be the wavelet associated to this fuzzy multiresolution analysis, then it generates a fuzzy orthonormal basis of $L^2([0,1], \beta([0,1]), \mu, \mathrm{F}(R))$.

Since in pratice, signals and images are considered of bounded support, they must be of finite resolution. Suppose that the 1D signal to be analyzed is defined on [0, 1] and contains N = 2^J points: then it can be represented by a vector $c_J = \{c_{J,k}\}_{k=0,\ldots\ldots,2^J-1}$

For the sake of simplicity, let us consider the case of periodic boundary conditions; this means that each of the coefficients $\{c_{j,k}\}_{k\in Z}$ *et* $\{d_{j,k}\}_{k\in Z}$ characterizing the projection of f(signal) into V_j and W_j is periodic with period 2^j : then $\forall k \in Z$,
$c_{j,k} = c_{j,k+2^j}$ and $d_{j,k} = d_{j,k+2^j}$.

Performing an orthogonal fuzzy wavelet transform of the signal f consist in decomposing it in $V_0 \underset{D}{\oplus} W_0 \underset{D}{\oplus} \ldots\ldots\ldots\ldots \underset{D}{\oplus} W_{J-1}$ and hence in finding the coefficients of its projection on V_0 on the one hand, and on each of the W_j , j = 0,........., J - 1 on the other hand.

Therefore f belongs to V_J means to: $f(t) = \sum_{k=0}^{2^J-1} c_{Jk} \ \phi_{Jk}(t)$.

ALGORITHM

Our algorithm, inspired by (Le Cadet, 2004), proceeds in 2 steps: analysis and synthesis.

Analysis:

The analysis step consists in finding, from these 2^J coefficients c_{Jk} , representing f on V_J , the 2^J coefficients c_{0k} and d_{jk} representing f on $V_0 \underset{D}{\oplus} W_0 \underset{D}{\oplus} \ldots\ldots\ldots\ldots \underset{D}{\oplus} W_{J-1}$.

Note the fact that $V_j = V_{j-1} \underset{D}{\oplus} W_{j-1}$.

Decomposing $\phi_{j-1,k}$ on V_j , we have :

$$\phi_{j-1,k} = \sum_{n=0}^{2^j-1} \left\langle \phi_{j-1,k}\,,\,\phi_{j,n} \right\rangle \phi_{j,n} \tag{4}$$

According to (Wu, 2002; Diamond, 2000; Butnariu, 1989) a fuzzy function θ is strongly measurable if and only if θ_α^U and θ_α^L are measurable $\forall \alpha \in [0,1]$, and taking the change of variable $2^{j-1} t_1 - k = t$, we can write:

$$\left\langle \phi_{j-1,k}, \phi_{j,n} \right\rangle = \int_0^1 \left[2^{\frac{j-1}{2}} \phi_\alpha^U \left(2^{j-1}t-k\right) 2^{\frac{j}{2}} \phi_\alpha^U\left(2^j t-n\right) + 2^{\frac{j-1}{2}} \phi_\alpha^L\left(2^{j-1}t-k\right) 2^{\frac{j}{2}} \phi_\alpha^L \left(2^j t-n\right)\right] d\alpha$$

$$= \int_0^1 \left[\frac{1}{\sqrt{2}} \phi_\alpha^U(t)\, \phi_\alpha^U\left(2t+2k-n\right) 2^j + \frac{1}{\sqrt{2}} \phi_\alpha^L(t)\ \phi_\alpha^L\left(2t+2k-n\right) 2^j \right] d\alpha$$

$$= \int_0^1 \left[\frac{2}{\sqrt{2}} \phi_\alpha^U(t)\, \phi_\alpha^U\left(2t+2k-n\right) + \frac{2}{\sqrt{2}} \phi_\alpha^L(t)\ \phi_\alpha^L\left(2t+2k-n\right) \right] d\alpha \quad \text{for j fixed at 1}$$

$$= \int_0^1 \left[\sqrt{2}\, \phi_\alpha^U(t)\, \phi_\alpha^U\left(2t-n+2k\right) + \sqrt{2}\, \phi_\alpha^L(t)\ \phi_\alpha^L\left(2t-n+2k\right) \right] d\alpha$$

$$= \int_0^1 \sqrt{2} \left[\phi_\alpha^U(t)\, \phi_\alpha^U\left(2t-n+2k\right) + \phi_\alpha^L(t)\ \phi_\alpha^L\left(2t-n+2k\right) \right] d\alpha$$

$$= h_{n-2k} \tag{5}$$

Therefore, (4) implies that :

$$\phi_{j-1,k} = \sum_{n=0}^{2^j-1} h_{n-2k}\, \phi_{j,n} \tag{6}$$

Calculating the scalar product of *f* with the vectors of each member of this equality, we have :

$$c_{j-1,k} = \sum_{n=0}^{2^j-1} c_{j,n}\, h_{n-2k}\,, \quad \forall k = 0,\dots\dots,2^{j-1}-1$$

Similarly, decomposing $\psi_{j-1,k}$ onto V_j , we have :

$$\psi_{j-1,k} = \sum_{n=0}^{2^j-1} \left\langle \psi_{j-1,k}\,,\,\phi_{j,n} \right\rangle \phi_{j,n}$$

As in (5), the change of variable $2^{j-1} t_1 - k = t$ proves that :

$$\left\langle \psi_{j-1,k}, \phi_{j,n} \right\rangle = \int_0^1 \sqrt{2} \left[\psi_\alpha^U(t)\, \phi_\alpha^U\left(2t-n+2k\right) + \psi_\alpha^L(t)\ \phi_\alpha^L\left(2t-n+2k\right) \right] d\alpha$$

$$= g_{n-2k} \tag{7}$$

And so $\psi_{j-1,k} = \sum_{n=0}^{2^j-1} g_{n-2k}\, \phi_{j,n}$ (8)

Taking the scalar product of *f* with each member of (8), we get :

$$d_{j-1,k} = \sum_{n=0}^{2^j-1} c_{j,n}\, g_{n-2k}\,, \quad \forall k = 0,\dots\dots,2^{j-1}-1$$

The equalities given by $c_{j-1,k}$ and $d_{j-1,k}$ can also be expressed in terms of circular convolution of period 2^j $\left(c_{j-1} = \left\{c_{j-1,k}\right\}_{k=0,\dots\dots,2^{j-1}-1}, d_{j-1}, c_j, d_j, h,\ g \textit{ sont ici des vecteurs}\right)$.

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Convolution - Decimation

From the discussion above, we obtain the following result:

$$c_{j-1}[k]=\left(c_j * \bar{h}\right)[2k] \qquad \forall k=0,\ldots\ldots,2^{j-1}-1 \tag{9}$$

$$d_{j-1}[k]=\left(c_j * \bar{g}\right)[2k] \qquad \forall k=0,\ldots\ldots,2^{j-1}-1 \tag{10}$$

with $\bar{h}[n]=h[-n]$ *et* $\bar{g}[n]=g[-n]$.

This result can be interpreted as follows:
h is a low-pass filter, which will smooth the coordinates keeping the low frequencies and *g* is a high-pass filter, which will select the details and give the high frequencies of the signal.

The fuzzy wavelet transform is obtained as an iteration of two operations: the data (initially, the vector c_J), are convolved by the filters h and g. The result of these two convolutions is that, only those of even index are kept, the other one is eliminated. This is the decimation.
The resulting vector c_{J-1} is used as a new starting point, and the vector d_{J-1} is stored.
c_{J-1} and d_{J-1} are of size 2^{J-1} while c_J is of size 2^J .
Figure 1 summarises this procedure.

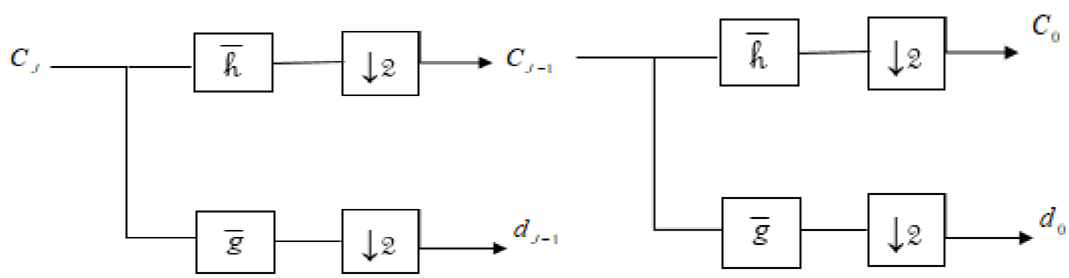

Figure 1: Analysis: filter bench

SYNTHESIS:

The synthesis is the opposite step of the analysis: from the wavelet coefficients, and thus from the vector data $\left[c_{00}, \{d_{jk}\}_{j=0,\ldots\ldots\ldots,J-1,k=0,\ldots\ldots\ldots,2^j-1}\right]$, we obtain
$c_J=\left[\left(c_{Jk}\right)_{k=0,\ldots\ldots\ldots,N-1}\right]$.
Indeed W_{j-1} is the orthogonal complement of V_{j-1} in V_j, the union of their bases $\{\psi_{j-1,n}\}_{n\in Z}$ and $\{\phi_{j-1,n}\}_{n\in Z}$ is a fuzzy orthonormal basis of V_j .
Therefore, any $\phi_{j,k}$ can be decomposed into this base by:

$$\phi_{j,k}=\sum_{n=0}^{2^{j-1}-1}\left\langle\phi_{j,k},\,\phi_{j-1,n}\right\rangle\phi_{j-1,n}+\sum_{n=0}^{2^{j-1}-1}\left\langle\phi_{j,k},\,\psi_{j-1,n}\right\rangle\psi_{j-1,n}\ .$$

Using (5) and (7) in this equality, we obtain :

$$\phi_{j,k}=\sum_{n=0}^{2^{j-1}-1}h_{k-2n}\,\phi_{j-1,n}+\sum_{n=0}^{2^{j-1}-1}g_{k-2n}\,\psi_{j-1,n}\ .$$

Taking the scalar product of *f* with each member of this equality, we find :

$$c_{jk} = \sum_{n=0}^{2^{j-1}-1} c_{j-1,n} \,.\, h_{k-2n} + \sum_{n=0}^{2^{j-1}-1} d_{j-1,n} \,.\, g_{k-2n} \quad .$$

This can also be written in vector form as $x_n^* = x_p \;\; si \;\; n = 2p$, $x_n^* = 0 \;\; si \;\; n = 2p+1$:

$$c_j\,[k\,] = \left(c_{j-1}^* * h\right)[k\,] + \left(d_{j-1}^* * g\right)[k] \qquad (11)$$

In other words, at each step, we double the size of c_j and d_j (on sampling) by interposing zeros between the coefficients of consecutively even index, then we convolve them with the filters *h* and *g*, and add the two terms.
Figure 2 illustrates this procedure

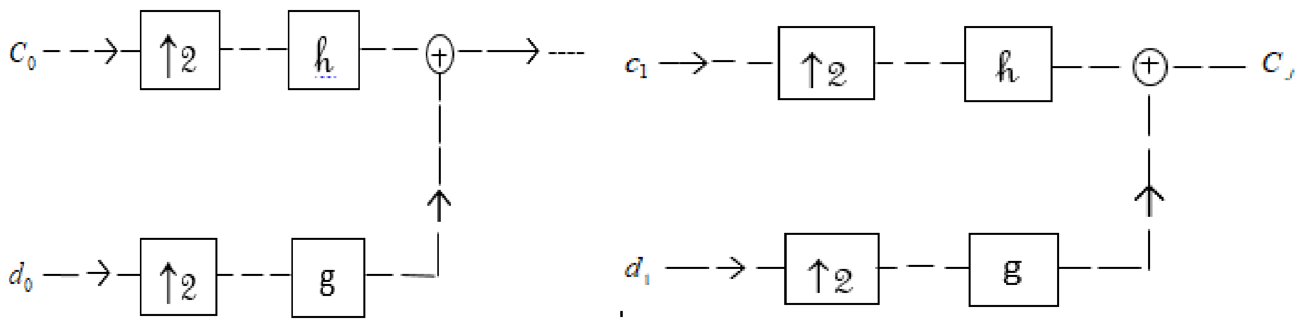

Figure 2: Synthesis: filter bench

Generalization to two dimensions

The general definition of a fuzzy multiresolution analysis of $L^2([0,1]^2, \beta(R^2), \vec{\mu}, \; \mathrm{F}(R))$ is similar to that given in the one-dimensional case.
It is sufficient to consider fuzzy functions defined on R^2 and no longer on R.
For this purpose, we need the concept of fuzzy tensor product.

Tensor product in $L^2([0,1]^2, \beta(R^2), \vec{\mu}, \; \mathrm{F}(R))$

Tensor products, in both the classical and fuzzy cases, are used to extend one-dimensional signal spaces to multi-dimensional signal spaces.

Definition 1.11
Let A and B be fuzzy sets.
We define the product [A. B] using α – cuts as follows:

$$[A.B]_\alpha = \{z : \alpha \in [0,1], \exists x \in A_\alpha , y \in B_\alpha \;\; avec \; z = x\, y\} \;.$$

Note that $A \underset{T}{\otimes} B(x) = A(x)\; B(x)$,

$$\left[A \underset{T}{\otimes} B(x)\right]_\alpha = [A(x)\; B(x)]_\alpha$$
$$= \{z : \; \exists x_0 \in A_\alpha , y_0 \in B_\alpha \;\; with \; z = x_0 \;\; y_0\}$$

Two-dimensional fuzzy multi-resolution analysis

Consider a sequence of nested fuzzy spaces $\{V_j\}_{j\in Z}$ defining a one-dimensional fuzzy multiresolution analysis of $L^2([0,1],\beta(R),\vec{\mu},\ \mathrm{F}(R))$.

Theorem 1.12

$\{\vartheta_j = V_j \underset{T}{\otimes} V_j\}_{j\in Z}$ defines a fuzzy multiresolution analysis of $L^2([0,1]^2,\beta(R^2),\vec{\mu},\ \mathrm{F}(R))$.

Proof

Since $\forall$ A, B fuzzy sets, we can define the product [A. B] using α-cuts as follows:
$[A.B]_\alpha = \{z : \alpha \in [0,1], \exists x \in A_\alpha, y \in B_\alpha\ avec\ z = x\ y\}$.
Since $A \underset{T}{\otimes} B(x) = A(x)\ B(x)$,

$$\left[A \underset{T}{\otimes} B(x)\right]_\alpha = [A(x)\ B(x)]_\alpha$$
$$= \{z :\ \exists x_0 \in A_\alpha, y_0 \in B_\alpha\ with\ z = x_0\ y_0\}$$

If we also consider $A_\alpha \subset A_{1\alpha}\ et\ B_\alpha \subset B_{1\alpha}$, we have : $A \subset A_1\ and\ B \subset B_1$.
Hence, taking $z = x_0\ y_0 \in (A\ B)_\alpha\ tel\ que\ x_0 \in A_\alpha, y_0 \in B_\alpha$, we also have: $x_0 \in A_{1\alpha}\ and\ y_0 \in B_{1\alpha}$.
For example, $z \in (A_1\ B_1)_\alpha$.
Therefore, posing H and H_1 by $A \underset{T}{\otimes} B\ et\ A_1 \underset{T}{\otimes} B_1$ respectively, we have :
$H = A \underset{T}{\otimes} B \subset H_1 = A_1 \underset{T}{\otimes} B_1$.
If $f \in H\ then\ \exists\ f_\alpha \in H_\alpha\ with\ f_\alpha(t) = \xi_\alpha(t)\,\eta_\alpha(t)\ and\ f_\alpha(2t) = \xi_\alpha(2t)\,\eta_\alpha(2t)$.
And let $\vartheta_j = V_j \underset{T}{\otimes} V_j$. It is clear that if $V_j \subset V_{j+1}$ then the first property below holds:

1) $\vartheta_j \subset \vartheta_{j+1}$.
2) Let $f : R \to \mathrm{F}(R) \times \mathrm{F}(R)$
$\quad t \to \xi(t).\eta(t)$

$$f_\alpha(t) = [\xi(t).\eta(t)]_\alpha = \xi_\alpha(t).\,\eta_\alpha(t)$$
If $\xi_\alpha(t) \in V_j^\alpha \Leftrightarrow \xi_\alpha(2t) \in V_{j+1}^\alpha$ $then$ $f_\alpha(t) \in \vartheta^\alpha_j \Leftrightarrow f_\alpha(2t) \in \vartheta^\alpha_{j+1}$.

$Hence$ $f(t) \in \vartheta_j \Leftrightarrow f(2t) \in \vartheta_{j+1}$.

3)
If $\xi_\alpha(t) \in V_0^\alpha \Leftrightarrow \xi_\alpha(t-k) \in V_0^\alpha$
$then$ $f_\alpha(t) \in \vartheta_0{}^\alpha \Leftrightarrow f_\alpha(t-k) \in \vartheta_0{}^\alpha$ that is $f(t) \in \vartheta_0 \Leftrightarrow f(t-k) \in \vartheta_0$.

Note that :

(i) $[\bigcap_{j=-N}^{N} \vartheta_j]^\alpha = \bigcap_{j=-N}^{N} \vartheta_J{}^\alpha$

(ii) $[\bigcup_{j=-N}^{N} \vartheta_i]^\alpha = \bigcup_{j=-N}^{N} \vartheta_j{}^\alpha$

5)

Starting from (ii), we have :

Firstly,

$$\lim_{N\to+\infty} \bigcup_{j=-N}^{N} \vartheta_j{}^\alpha = \bigcup_{j=-\infty}^{+\infty} \vartheta_j{}^\alpha$$

and on the other hand,

$$\lim_{N\to+\infty} (\bigcup_{j=-N}^{N} \vartheta_j)^\alpha = (\lim_{N\to\infty} \bigcup_{-N}^{N} \vartheta_j)^\alpha .$$

Hence

$$\lim_{N\to+\infty} \bigcup_{j=-N}^{N} \vartheta_j = \bigcup_{j=-\infty}^{+\infty} \vartheta_j .$$

And since $\vartheta_j \subset \vartheta_{j+1}$, we deduce that $\lim_{j\to+\infty} \vartheta_j = \overline{\bigcup_{j=-\infty}^{+\infty} \vartheta_j}$.

4) Since $V_j \subset V_{j+1}$, *we have* : $V_j^\alpha \subset V_{j+1}^\alpha$. Hence $\vartheta_j{}^\alpha$ form decreasing nested intervals when $j \to -\infty$ that is $\vartheta^\alpha{}_{-(j+1)} \subset \vartheta^\alpha{}_{-j}$.

Therefore : $\bigcap_{j=-\infty}^{\infty} \vartheta_j{}^\alpha = \{0\}$ and $\lim_{j\to+\infty} \vartheta_j{}^\alpha = \{0\} = \bigcap_{j=-N}^{N} \vartheta_j{}^\alpha = (\lim_{N\to\infty} \bigcap_{-N}^{N} \vartheta_j)^\alpha$ by reasoning similarly to 5) and using (i).

Hence : $\lim_{j\to+\infty} \vartheta_j = \bigcap_{j=-\infty}^{\infty} \vartheta_j = \chi_{\{0\}}$.

To complete the construction of the fuzzy multiresolution analysis, we prove the following theorem:

Theorem 1.13

Let θ_j and Φ_j be two bases of the fuzzy space V_j, then $\left\{\theta_j(t-n).\phi_j(k-m) = \theta_j \underset{T}{\otimes} \phi_j(t-n, k-m)\right\}_{(n,m)\in Z^2}$ is a Riesz basis of $\vartheta_j = V_j \underset{T}{\otimes} V_j$.

Proof

Indeed, θ_j and ϕ_j being two bases of V_j, $\theta_j \underset{T}{\otimes} \phi_j$ is a base of ϑ_j that is $\theta_j^\alpha \otimes \phi_j^\alpha$ is a base of $\vartheta_j{}^\alpha$.

By Theorem 1.4, $\theta_j^\alpha \otimes \phi_j^\alpha(t-n, k-m) = \theta_j^\alpha(t-n).\phi_j^\alpha(k-m)$ is a Riesz basis of $\vartheta_j{}^\alpha = V_j^\alpha \otimes V_j^\alpha$ if and only if $\exists\ 0 < C$ and $0 < D$ such that $\forall\, w \in [-\pi, \pi]$,

$$\frac{1}{D} \le \left| \hat{\theta}_j^\alpha(w+2k\pi) . \hat{\phi}_j^\alpha(w+2k\pi) \right|^2 \le \frac{1}{C} .$$

Hence, $\theta_j \underset{T}{\otimes} \phi_j$ is a Riesz basis of ϑ_j if and only if $\exists\ 0 < C$ and $0 < D$ such that $\forall\, w \in [-\pi, \pi]$,

$$\frac{1}{D} \le \left| \hat{\theta}_j(w+2k\pi) . \hat{\phi}_j(w+2k\pi) \right|^2 \le \frac{1}{C} .$$

Therefore, $\{\vartheta_j = V_j \underset{T}{\otimes} V_j\}_{j \in Z}$ generates a fuzzy multiresolution analysis of $L^2([0,1]^2, \beta(R^2), \vec{\mu}, \mathrm{F}(R))$.

Thus, the fast fuzzy wavelet transform algorithm presented in one dimension can be extended to two dimensions.
Consider for all scales 2^j and for all $n = (n_1, n_2)$:

$c_{j,n} = \langle f, \phi^2_{j,n} \rangle$ and $d^k_{j,n} = \langle f, \psi^k_{j,n} \rangle$ for $1 \le k \le 3$ where

$$\Big\{ \phi^2_{j,n}(x) = \phi_{j,n_1}(x_1)\,\phi_{j,n_2}(x_2) = 2^j \phi(2^j x_1 - n_1)\,\phi(2^j x_2 - n_2)$$
$$= 2^j \int_0^1 \Big[\phi^U_\alpha(2^j x_1 - n_1)\,\phi^U_\alpha(2^j x_2 - n_2) + \phi^L_\alpha(2^j x_1 - n_1)\,\phi^L_\alpha(2^j x_2 - n_2) \Big] d\alpha \Big\}_{n \in Z^2}$$

(fuzzy orthonormal basis of ϑ_j)

and $\Big\{ \psi^k_{j,n}(x) = 2^j \psi^k(2^j x_1 - n_1, 2^j x_2 - n_2) \Big\}_{(j,n) \in Z^3}$

(fuzzy orthonormal basis of $L^2([0,1]^2, \beta(R^2), \vec{\mu}, \mathrm{F}(R))$)

with $\psi^1(x) = \phi(x_1)\,\psi(x_2) = \int_0^1 \Big[\phi^U_\alpha(x_1)\,\psi^U_\alpha(x_2) + \phi^L_\alpha(x_1)\,\psi^L_\alpha(x_2) \Big] d\alpha$,

$$\psi^2(x) = \psi(x_1)\,\phi(x_2) = \int_0^1 \Big[\psi^U_\alpha(x_1)\,\phi^U_\alpha(x_2) + \psi^L_\alpha(x_1)\,\phi^L_\alpha(x_2) \Big] d\alpha \quad ,$$

$$\psi^3(x) = \psi(x_1)\,\psi(x_2) = \int_0^1 \Big[\psi^U_\alpha(x_1)\,\psi^U_\alpha(x_2) + \psi^L_\alpha(x_1)\,\psi^L_\alpha(x_2) \Big] d\alpha$$

For any pair of one-dimensional filters y [m] and z [m], we can write yz [n] = y [n_1]. z[n_2], and $\bar{y}[m] = y[-m]$.
Consider h[m] and g[m] as two conjugate filters associated with the fuzzy wavelet ψ .

ALGORITHM

Decomposition

Wavelet coefficients at scale 2^j are calculated from c_j by convolution and two-dimensional separable subsampling.
The decomposition formulas are obtained by applying the one-dimensional convolution formulas given in (9) and (10) to the separable two-dimensional fuzzy wavelets and scaling functions for $n = (n_1, n_2)$:

$$c_{j-1}[n] = (c_j * \bar{h}\,\bar{h})[2n] \quad (12)$$

$$d^1_{j-1}[n] = (c_j * \bar{h}\,\bar{g})[2n] \quad (13)$$

$$d^2_{j-1}[n] = (c_j * \bar{g}\,\bar{h})[2n] \quad (14)$$

$$d^3_{j-1}[n] = (c_j * \bar{g}\,\bar{g})[2n] \quad (15)$$

This result can be interpreted as follows:
A separable two-dimensional convolution can be factored into the one-dimensional convolutions along the rows and columns of the fuzzy image.
As the factorisation is illustrated in Figure 3, these 4 convolution equations are obtained with only 6 one-dimensional convolution groups.
The lines of c_J are first convolved with $\bar{h}$ and $\bar{g}$ and sub-sampled by 2.

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Then, the columns of these two produced images are convoluted respectively with $\bar{h}$ and $\bar{g}$ and subsampled to give the 4 subsampled images c_{J-1} , d^1_{J-1} , d^2_{J-1} *et* d^3_{J-1}.
We can conclude as follows:
The fuzzy wavelet transform of the image c_L gives 3J + 1 fuzzy sub-images
$[\, c_{L-J} , \{d_j^{\,1} , d_j^{\,2} , d_j^{\,3} \}_{L-J\leq j<L} \,]$, (16)
calculated by iteration of the recursion relations (12) - (15) for L - J ≤ j < L ; J being the number of octaves (frequency band) considered for this decomposition.

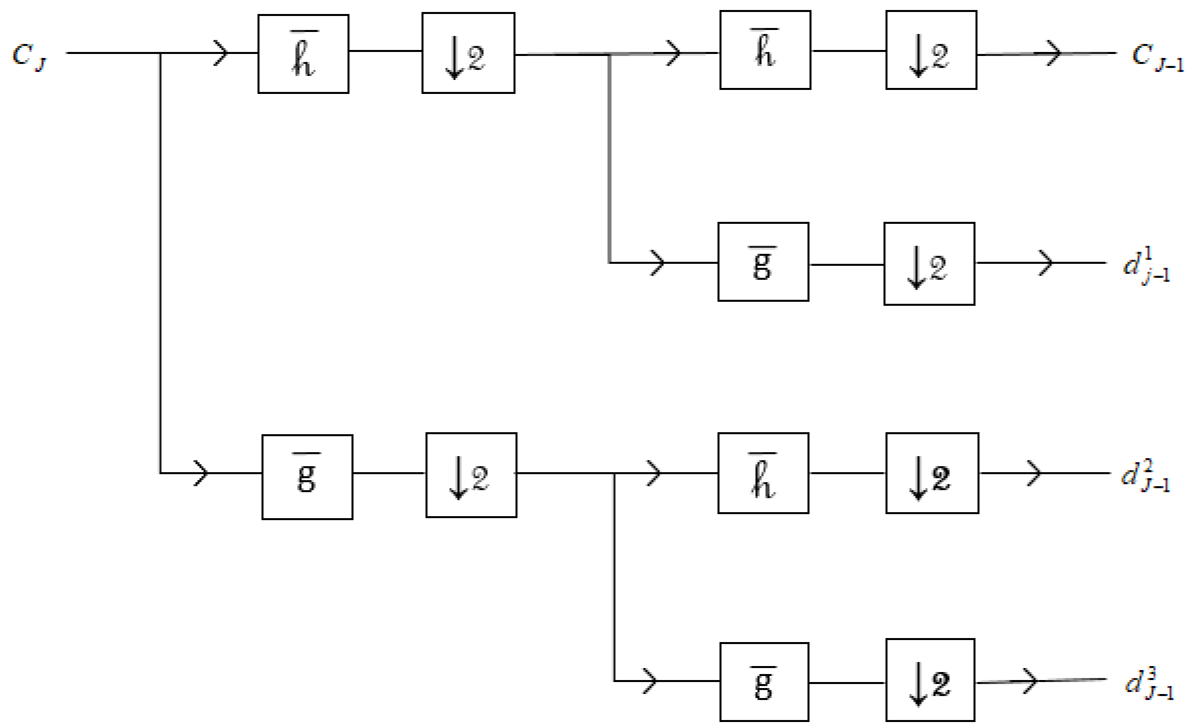

Figure 3: Decomposition of c_J into 6 convolution groups and one-dimensional sub-samples along the rows and columns of the blurred image.

RECONSTRUCTION

Let ỹ $[n]$ = ỹ $[n_1 , n_2]$ denote the fuzzy image of size two, obtained by inserting a row of zeros and a column of zeros between rows and even columns consecutively.
The approximation c_J is reconstructed from the coarse scale approximation c_{J-1} and the wavelet coefficients $d_{J-1}^{\,k}$ with separable two-dimensional convolutions derived from the one-dimensional reconstruction formula in (11).

The result is :

$$c_j[n] = (c_{j-1} * hh)[n] + (d^1_{j-1} * h\,g)[n] + (d^2_{j-1} * g\,h)[n] + (d^3_{j-1} * g\,g)[n] \qquad (17)$$

Therefore, the image c_L is reconstructed by the wavelet representation (17) for L - J ≤ j < L .
The 4 separable convolutions in (19) can also be factorised into 6 groups of one-dimensional convolutions along the rows and columns, as shown in Figure 4.

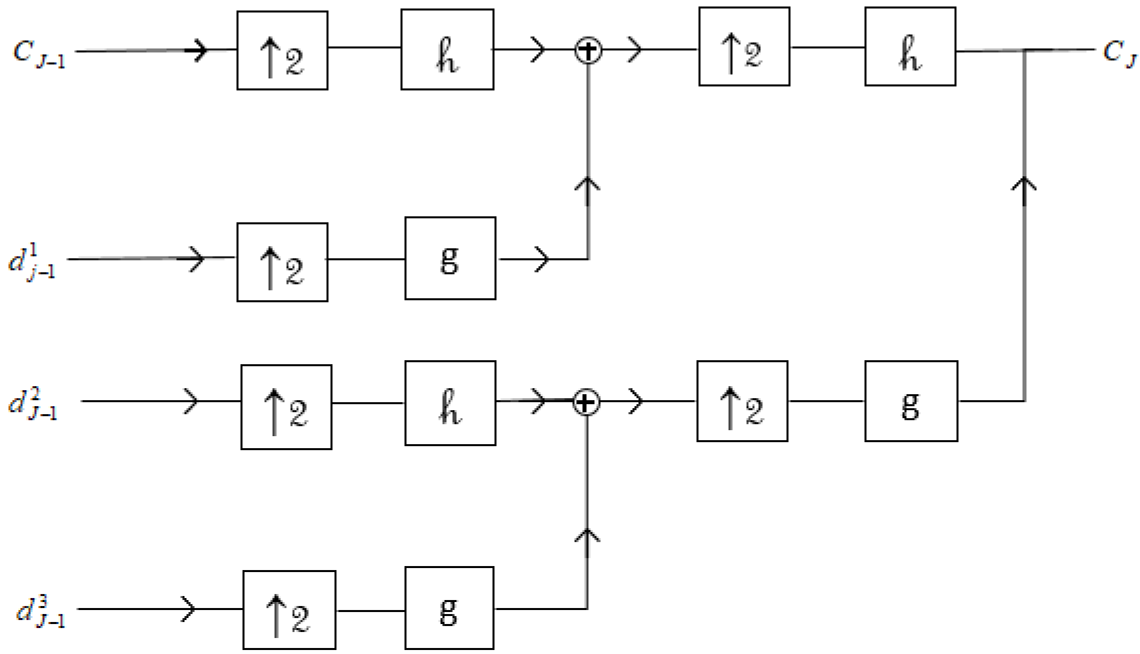

Figure 4: Reconstruction of c_J by inserting the zeros between the rows and columns of c_{J-1} and $d_{J-1}{}^k$, and filtering the result.

Discussions

Our results, mainly the definition and the proof of two – dimensional fuzzy multiresolution analysis as well as the construction of decomposition et reconstruction algorithms, constitute our major and original contribution.

CONCLUSION

The paper deals with new algorithm for decomposition and reconstruction of a fuzzy one- or two-dimensional fuzzy signal in fuzzy wavelets. In general case, decomposition and reconstruction of image follow a suitable procedure. For the decomposition: The image is divided into a low-resolution part, which can be described by a smaller number of bits than the original image, and a signal difference, which describes the difference between the low-resolution image and the real coded image.
The low-resolution image is in turn decomposed into another low-resolution image and a difference image, making more efficient coding possible. Repeating this decomposition several times, we obtain a hierarchical image. The resulting low-resolution image has only half the size of the original image.
The reduced image is then enlarged to the size of the original image. The result is a detailed image that has the same size as the original image. Similarly, fuzzy wavelet decomposition reduces the size of the fuzzy image at each stage and each fuzzy detail in these images is taken into account by the fuzzy wavelet coefficients which allow a good reconstruction of signal for better transmission. In this work, we have described, using orthonormal wavelet transform, algorithms for the decomposition and reconstruction in fuzzy wavelets for a one and two-dimensional signal by associated fuzzy multiresolution analyses.

CONFLICT OF INTEREST DISCLOSURE

The authors have disclosed that they have no potential conflicts of interest.

REFERENCES

Antoine, J. P., Murenzi, R., Vandergheynst, P., & Ali, S. T. (2008). *Two-dimensional wavelets and their relatives*. Cambridge University Press.

Bloch, I. (2015). Fuzzy sets for image processing and understanding. *Fuzzy sets and systems*, *281*, 280-291.

Butnariu, D., (1989). Measurability concepts for fuzzy mappings, Fuzzy Sets and Systems, 31, 77 – 82.

Chachi, J. (2018). On distribution characteristics of a fuzzy random variable. *Austrian Journal of Statistics*, *47*(2), 53-67.

Cognet, M. (2000), Algèbre linéaire, Bréal.

Daubechies, I. (1992). *Ten lectures on wavelets*. Society for industrial and applied mathematics.

De Barros, L. C., & Santo Pedro, F. (2017). Fuzzy differential equations with interactive derivative. *Fuzzy sets and systems*, *309*, 64-80.

Diamond, P., Kloeden, P., (2000). Metric topology of fuzzy numbers and fuzzy analysis. In: D. Dubois, H. Pride, et al.(eds.) Hand book fuzzy Sets Ser, vol.7, pp. 583 - 641. Kluwer Academic publishers, Dordrecht.

Feng, Y., L. Hu, H. Shu. (2001), The variance and covariance of fuzzy random variables and their Applications, Fuzzy Set Syst., 120, 487 – 497.

Gomes, L. T., de Barros, L. C., & Bede, B. (2015). *Fuzzy differential equations in various approaches*. Berlin: Springer.

Grifone, J. (2019). *Algèbre Linéaire 6E Édition*. Éditions Cépaduès.

Hesamian, G., & Ghasem Akbari, M. (2022). Testing hypotheses for multivariate normal distribution with fuzzy random variables. *International Journal of Systems Science*, *53*(1), 14-24.

Kumwimba, D. (2016). Analyse stochastique floue et application aux options financières : cas du Modèle de Blach – Scholes Flou, Thèse de Doctorat, Université de Kinshasa, RDC.

Lakshmikantham, V. & R.N. Mohapatra. (2003), Theory of Fuzzy Differential Equations and Inclusions, London EC 4P4EE, p. 14 – 15.

Le Cadet, O. (2004). Méthodes d'ondelettes pour la segmentation d'images. Applications à l'imagerie médicale et au tatouage d'images, Thèse de Doctorat, Institut National polytechnique (Grenoble), France.

Mallat, S. (1999). *A wavelet tour of signal processing*. Elsevier.

Mazandarani, M., & Xiu, L. (2021). A review on fuzzy differential equations. *IEEE Access*, *9*, 62195-62211.

Meyer, Y., Jaffard, S., & Rioul, O. (1987). L'analyse par ondelettes. *Pour la science*, *119*, 28-37.

Ohlan, R., & Ohlan, A. (2021). A bibliometric overview and visualization of fuzzy sets and systems between 2000 and 2018. *The Serials Librarian*, *81*(2), 190-212.

Perfilieva, I., (2006), Fuzzy transforms, Theory and applications, Fuzzy Sets and Systems, 157(8) 993-1023.

Sussner, P. (2016). Lattice fuzzy transforms from the perspective of mathematical morphology. *Fuzzy Sets and Systems*, *288*, 115-128.

Wu, J., (2002). An embedding Theorem for fuzzy numbers on Banach spaces and its applications, Fuzzy Sets and Systems 129, 57 - 63.